On Interconnections of "Mixed" Systems Using Classical Stability Theory*

Wynita M. Griggs†‡, S. Shravan K. Sajja†‡, Brian D. O. Anderson§ and Robert N. Shorten†

Abstract

In this paper, we derive stability results for large-scale interconnections of "mixed" linear, time-invariant systems using classical Nyquist arguments. We compare our results with Moylan and Hill [1]. Our results indicate that, if one relaxes assumptions on the subsystems in an interconnection from assumptions of passivity or small gain to assumptions of "mixedness," then the Moylan- and Hill-like conditions on the interconnection matrix become more stringent. Finally, we explore a condition for the stability of large-scale, time-varying interconnections of strictly positive real systems. This condition mirrors the condition obtained in [1] for time-invariant interconnections and is thus an extension of this work.

1 Introduction

A situation that inspires the study of "mixed" systems [2, 3] is one in which high frequency dynamics neglected for modelling purposes destroy the passivity properties of an otherwise passive system. These unmodelled dynamics will always be present in a real system. As such, the passivity theorem alone may not be adequate to show that the stability of the system interconnection is guaranteed [4]. The book [5], see also [6] and [7], described tools for establishing the stability of adaptive systems of the type examined in [4]; that is, where passivity-type properties hold only for low frequency signals.

*This work was supported by SFI grant 07/IN.1/I901 and will be presented at the 2012 American Control Conference in Montréal, Canada.

†W. Griggs, S. Sajja and R. Shorten are with the Hamilton Institute, National University of Ireland, Maynooth, Co. Kildare, Ireland. Corresponding author: Wynita M. Griggs. Phone: +353-(0)1-7086100. Fax: +353-(0)1-7086269. Email: wynita.griggs@nuim.ie

‡Joint first authors.

§B. Anderson is with: (i) Research School of Engineering, ANU College of Engineering and Computer Science, RSISE Building 115, Australian National University, Canberra ACT 0200, Australia; and (ii) National ICT Australia Limited, Locked Bag 8001, Canberra ACT 2601, Australia.

"Mixed" linear, time-invariant (LTI) systems, as defined in [8], are systems that combine notions of passivity and small gain behaviour in a certain manner. Roughly speaking, "mixed" systems exhibit small gain behaviours over frequency bands where passivity behaviour is violated. Hence, "mixed" systems formalise a notion that engineers have intuitively held for a long time: that keeping feedback-loop gain small at those frequencies where passivity is violated will avoid destabilisation of high frequency dynamics. A test for determining whether multi-input, multi-output (MIMO), LTI systems are "mixed" was introduced in [8].

Independently, the study of the stability of large-scale interconnections of systems is of increasing importance. Some works on this topic include [1, 9–11].

In this paper, we apply classical Nyquist techniques to give stability results for interconnections of "mixed" LTI systems; see Sections 3 and 4. Previous work in this direction appeared in [12, 13]. Our work goes beyond [13] in a number of ways. First, we present more detailed Nyquist arguments, based essentially on a Lyapunov argument. Secondly and most importantly, we utilise the techniques to obtain new sufficient conditions for the stability of large-scale interconnections of "mixed" systems. Our large-scale interconnection results suggest that, as one relaxes the assumptions on the transfer function matrices of the systems, eg: from assumptions of passivity to assumptions of "mixedness," the Moylan- and Hill-like conditions [1] on the interconnection matrix become more severe.

This paper also corrects an error in Theorems 1, 6, 3 and 9 of [2, 3, 14] and [15], respectively. Determining bounded input, bounded output (BIBO) and finite-gain stability of interconnections of "mixed" LTI systems in a dissipative systems framework was the concern of these works. Roughly speaking, a system that produces a bounded output for any bounded input is said to be BIBO stable. The issue with the aforementioned results, however, is that the system output was assumed to be bounded *a priori*. In effect, the works indicate the existence of a bound on the output in terms of the input; but where BIBO stability is already assumed. Our present treatment of "mixed" LTI system interconnections via Nyquist techniques provides an approach for deriving the originally desired BIBO stability results.

Finally, in Section 5, we explore the stability of large-scale, time-varying interconnections of single-input, single-output (SISO), strictly proper, strictly positive real (SPR) systems. We derive a condition that guarantees the existence of a Lyapunov function for the interconnected system. Particularly, we show that, by replacing passivity with SPRness as an assumption on the subsystems in a time-varying interconnection, the classic result [1, Theorem 4] extends in such a way that the condition on the interconnection matrix H now becomes that there exists a diagonal matrix $Q > 0$ such that $H(t)^T Q + QH(t) \geq 0$ for all time $t \geq 0$. This follows from the Kalman-Yakubovich-Popov (KYP) lemma. We conclude the paper in Section 6 with a summary of our results and some directions for future research.

Notation

The notation $\Re[s]$ will be used to denote the real part of a complex number s. The conjugate of a complex number $s = a + jb$, where a, b are real and $j^2 = -1$, will be denoted by $\bar{s} := a - jb$. For a nonsingular matrix A, $A^{-*} := (A^{-1})^* = (A^*)^{-1}$, where A^* denotes the conjugate transpose of A. The largest and smallest singular values of a matrix A will be denoted by $\bar{\sigma}(A)$ and $\underline{\sigma}(A)$, respectively. For a transfer function matrix G of a LTI system, $G^*(j\omega) := [G(j\omega)]^*$. $\mathcal{R}$ denotes the set of proper, real-rational transfer function matrices. $\mathcal{L}_\infty$ is a Banach space of matrix- (or scalar-) valued functions that are essentially bounded on $j\mathbb{R}$ with norm $\|G\|_\infty := \text{ess}\sup_{\omega\in\mathbb{R}} \bar{\sigma}(G(j\omega))$. $\mathcal{RL}_\infty := \mathcal{R} \cap \mathcal{L}_\infty$ consists of all proper, real-rational transfer function matrices with no poles on the imaginary axis. The Hardy space $\mathcal{H}_\infty$ is the closed subspace of $\mathcal{L}_\infty$ with functions that are analytic and bounded in the open right-half plane (RHP). In other words, $\mathcal{H}_\infty$ is the space of transfer function matrices of stable, LTI, continuous-time systems. $\mathcal{RH}_\infty := \mathcal{R} \cap \mathcal{H}_\infty$ consists of all proper, real-rational transfer function matrices with no poles in the closed RHP.

2 Definitions and Preliminary Results

Before deriving the main results of the paper, we establish a number of definitions. Consider a causal system with square transfer function matrix $M \in \mathcal{RH}_\infty$. Suppose that $a, b, c, d \in \mathbb{R}$.

Definition 1. [8] A causal system with square transfer function matrix $M \in \mathcal{RH}_\infty$ is said to be input and output strictly passive over a frequency interval $[a, b]$, $(-\infty, c]$, $[d, \infty)$ or $(-\infty, \infty)$ if there exist $k, l > 0$ such that

$$-kM^*(j\omega)M(j\omega) + M^*(j\omega) + M(j\omega) - lI \geq 0$$

for all $\omega \in [a, b]$, $(-\infty, c]$, $[d, \infty)$ or $(-\infty, \infty)$, respectively.

We will say that the system is input strictly passive over a frequency interval if Definition 1 is satisfied with $k = 0$; output strictly passive over a frequency interval if the definition is satisfied with $l = 0$; and passive over a frequency interval if it is satisfied with $k = l = 0$. Note that any $M(j\omega)$ satisfying Definition 1 over the frequency interval $(-\infty, c]$, $[d, \infty)$ or $(-\infty, \infty)$ must be such that $\lim_{\omega\to\pm\infty} \lambda_i[M^*(j\omega) + M(j\omega)] = c_{p_i} > 0$ for all i, where $\lambda_i \in \mathbb{R}$ denotes the ith eigenvalue of the Hermitian matrix $M^*(j\omega) + M(j\omega)$. Then $\lim_{\omega\to\pm\infty} \det[M^*(j\omega) + M(j\omega)] \neq 0$.

Definition 2. [8] Define the system gain over the frequency interval $[a, b]$, $(-\infty, c]$, $[d, \infty)$ or $(-\infty, \infty)$ as

$$\epsilon := \inf\{\bar{\epsilon} \in \mathbb{R}_+ : -M^*(j\omega)M(j\omega) + \bar{\epsilon}^2 I \geq 0 \text{ for all } \omega \in [a, b],\ (-\infty, c],\ [d, \infty) \text{ or } (-\infty, \infty), \text{ respectively}\}.$$

The causal system with transfer function matrix $M \in \mathcal{RH}_\infty$ is said to have a gain of less than one over the frequency interval $[a, b]$, $(-\infty, c]$, $[d, \infty)$ or $(-\infty, \infty)$, respectively, if $\epsilon < 1$.

For any system satisfying Definition 2 with gain of less than one over the frequency interval $(-\infty, c]$, $[d, \infty)$ or $(-\infty, \infty)$ it must hold that $\lim_{\omega\to\pm\infty} \lambda_i[-M^*(j\omega)M(j\omega)+I] = c_{s_i} > 0$ for all i, where $\lambda_i \in \mathbb{R}$ denotes the ith eigenvalue of the Hermitian matrix $-M^*(j\omega)M(j\omega)+I$. Then $\lim_{\omega\to\pm\infty} \det[-M^*(j\omega)M(j\omega) + I] \neq 0$. We now define a "mixed" system similarly to [8].

Definition 3. A causal system with square transfer function matrix $M \in \mathcal{RH}_\infty$ is said to be "mixed" if, for each frequency $\omega \in \mathbb{R} \cup \{\pm\infty\}$: either (i) $-kM^*(j\omega)M(j\omega) + M^*(j\omega) + M(j\omega) - lI \geq 0$; and/or (ii) $-M^*(j\omega)M(j\omega) + \epsilon^2 I \geq 0$. The constants $k, l > 0$ and $\epsilon < 1$ are independent of ω.

An example of a "mixed" system is a system with transfer function

$$m(s) = \frac{3}{(s+1)(s+2)}$$

and Nyquist diagram as depicted in Figure 1. From the Nyquist diagram, it is clear that there exists a frequency Ω such that, over the frequency band $[-\Omega, \Omega]$, Property (i) of Definition 3 holds and, over the frequency bands $[-\infty, -\Omega]$ and $[\Omega, \infty]$, Property (ii) of the definition is satisfied. (Note that $\Re[m(j\omega)] = \frac{1}{2}[m^*(j\omega) + m(j\omega)]$ and $|m(j\omega)|^2 = m^*(j\omega)m(j\omega)$.)

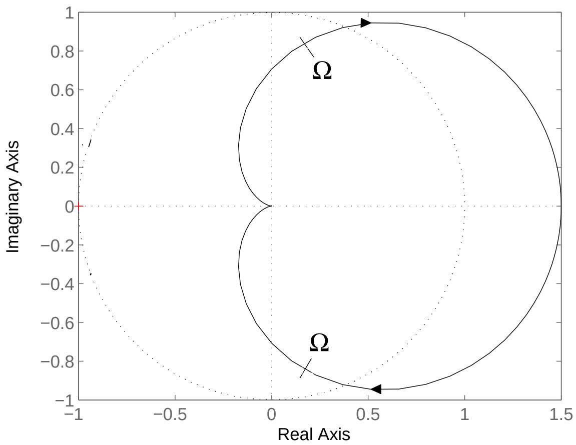

Figure 1: Nyquist diagram of $m(s)$.

We will also require the following preliminary results.

Lemma 1. *Suppose that $G_1 \in \mathcal{RL}_\infty$ and $G_2 \in \mathcal{RL}_\infty$. Suppose further that, at some $\omega \in \mathbb{R}\cup\{\pm\infty\}$, $G_1^*(j\omega)+G_1(j\omega) > 0$ and $G_2^*(j\omega)+G_2(j\omega) \geq 0$. Then* $\det[I+G_1(j\omega)G_2(j\omega)] \neq 0$.

Proof. Since $G_1^*(j\omega) + G_1(j\omega) > 0$, $\Re[\lambda_i[G_1(j\omega)]] > 0$ $\forall i$ (where $\lambda_i[\cdot]$ denotes the ith eigenvalue) [16, Theorem 1 of Section 13.1] and so $G_1(j\omega)$ is nonsingular. Then $G_1^{-*}(j\omega) +$

$G_1^{-1}(j\omega) > 0$ since $G_1^*(j\omega) + G_1(j\omega)$ and $G_1^{-1}(j\omega) + G_1^{-*}(j\omega)$ are Hermitian congruent [17, page 415] [18, Section V.3]. Then $G_1^{-*}(j\omega) + G_2^*(j\omega) + G_1^{-1}(j\omega) + G_2(j\omega) > 0$. Hence $\Re[\lambda_i[G_1^{-1}(j\omega)+G_2(j\omega)]] > 0\ \forall i$ and so $\det[G_1^{-1}(j\omega)+G_2(j\omega)] \neq 0$. Then $\det[I+G_1(j\omega)G_2(j\omega)] \neq 0$ since $\det[I + G_1(j\omega)G_2(j\omega)] = \det[G_1(j\omega)]\det[G_1^{-1}(j\omega) + G_2(j\omega)]$ and $G_1(j\omega)$ is nonsingular. □

Letting $G_1 = I$ and setting $G := G_2$ in the above lemma statement gives the following corollary. (Alternatively, we can set $G := G_1$ and let $G_2 = I$ to obtain a version of the corollary containing a strict inequality.)

Corollary 2. *Suppose that $G \in \mathcal{RL}_\infty$ and that, at some $\omega \in \mathbb{R}\cup\{\pm\infty\}$, $G^*(j\omega)+G(j\omega) \geq 0$. Then $\det[I + G(j\omega)] \neq 0$.*

Versions of the next corollary can be found in [19, Lemma 7 of Section VI.10] and [20, Theorem 2.3.4].

Corollary 3. *Suppose that $G \in \mathcal{RL}_\infty$ and that, at some $\omega \in \mathbb{R}\cup\{\pm\infty\}$, $G^*(j\omega)+G(j\omega) \geq 0$. Let $S(j\omega) := (G(j\omega) - I)(I + G(j\omega))^{-1}$. Then $-S^*(j\omega)S(j\omega) + I \geq 0$.*

Proof. From Corollary 2, $\det[I + G(j\omega)] \neq 0$. Then

$$\begin{aligned} &2(I + G(j\omega))^{-*}[G^*(j\omega) + G(j\omega)](I + G(j\omega))^{-1} \\ &= (I + G(j\omega))^{-*}[(I + G(j\omega))^*(I + G(j\omega)) - (G(j\omega) - I)^*(G(j\omega) - I)](I + G(j\omega))^{-1} \\ &= I - (I + G(j\omega))^{-*}(G(j\omega) - I)^*(G(j\omega) - I)(I + G(j\omega))^{-1} \\ &= I - S^*(j\omega)S(j\omega). \end{aligned}$$

Since $G^*(j\omega) + G(j\omega)$ and $I - S^*(j\omega)S(j\omega)$ are Hermitian congruent, $-S^*(j\omega)S(j\omega) + I \geq 0$. □

An extension to Lemma 1 is given below.

Lemma 4. *Suppose that $G_1 \in \mathcal{RL}_\infty$ and $G_2 \in \mathcal{RL}_\infty$. Suppose further that, at some $\omega \in \mathbb{R} \cup \{\pm\infty\}$, $G_1^*(j\omega) + G_1(j\omega) > G_1^*(j\omega)KG_1(j\omega)$ and $G_2^*(j\omega) + G_2(j\omega) \geq -K$, where K is a real-symmetric, positive semidefinite matrix. Then $\det[I + \frac{1}{\kappa}G_1(j\omega)G_2(j\omega)] \neq 0$ for any $\kappa \geq 1$, where $\kappa \in \mathbb{R}$.*

Proof. Since $G_1^*(j\omega) + G_1(j\omega) > G_1^*(j\omega)KG_1(j\omega) \geq 0$, $G_1^*(j\omega) + G_1(j\omega) > 0$ and so $\Re[\lambda_i[G_1(j\omega)]] > 0\ \forall i$ (where $\lambda_i[\cdot]$ denotes the ith eigenvalue) [16, Theorem 1 of Section 13.1]. Then $G_1(j\omega)$ is nonsingular and hence $G_1^{-*}(j\omega) + G_1^{-1}(j\omega) > K$ since $G_1^*(j\omega) + G_1(j\omega) - G_1^*(j\omega)KG_1(j\omega)$ and $G_1^{-1}(j\omega) + G_1^{-*}(j\omega) - K$ are Hermitian congruent. Moreover, $\kappa(G_1^{-*}(j\omega) + G_1^{-1}(j\omega)) > K$ for any $\kappa \geq 1$. Then $\kappa G_1^{-*}(j\omega) + \kappa G_1^{-1}(j\omega) - K + G_2^*(j\omega) + G_2(j\omega) + K = \kappa G_1^{-*}(j\omega) + G_2^*(j\omega) + \kappa G_1^{-1}(j\omega) + G_2(j\omega) > 0$. Hence $\Re[\lambda_i[\kappa G_1^{-1}(j\omega) + G_2(j\omega)]] > 0\ \forall i$ and so $\det[\kappa G_1^{-1}(j\omega)+G_2(j\omega)] \neq 0$. Then $\det[I + \frac{1}{\kappa}G_1(j\omega)G_2(j\omega)] \neq 0$ since $\det[I + \frac{1}{\kappa}G_1(j\omega)G_2(j\omega)] = \det[\frac{1}{\kappa}G_1(j\omega)]\det[\kappa G_1^{-1}(j\omega) + G_2(j\omega)]$ and $\frac{1}{\kappa}G_1(j\omega)$ is nonsingular. □

Lastly, since our aim is to deduce the stability of interconnections of "mixed" systems using arguments based on classical Nyquist techniques, we state a MIMO version of the Nyquist stability theorem.

Theorem 5. *[21, Theorem 5.8] [22, Remark 4 of Section 4.9.2] Consider the feedback interconnection of systems depicted in Figure 2. Suppose that $G_1 \in \mathcal{RH}_\infty$, $G_2 \in \mathcal{RH}_\infty$ and that the system interconnection is well-posed. Then the feedback-loop is stable if and only if the Nyquist plot of* $\det[I+G_1(j\omega)G_2(j\omega)]$ *for $-\infty \leq \omega \leq \infty$ does not make any encirclements of the origin.*

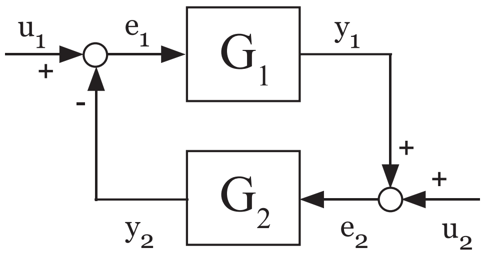

Figure 2: A negative feedback interconnection.

In the above theorem, well-posedness and stability are defined in the sense of [21, Sections 5.2 and 5.3]. Note, also, the following observations concerning the Nyquist plot of $\det[I + G_1(j\omega)G_2(j\omega)]$.

Observation 1. The Nyquist plot of $\det[I + G_1(j\omega)G_2(j\omega)]$ belongs to a family of Nyquist plots of $\det[I + \frac{1}{\kappa}G_1(j\omega)G_2(j\omega)]$, where $\kappa \in [1, \infty)$.

Observation 2. Each Nyquist plot of $\det[I+\frac{1}{\kappa}G_1(j\omega)G_2(j\omega)]$ is symmetrical about the real axis of the complex plane, where $\kappa \in [1, \infty)$.[1]

Observation 3. As κ and ω vary continuously, the point in the complex plane on which the Nyquist plot of $\det[I + \frac{1}{\kappa}G_1(j\omega)G_2(j\omega)]$ lies varies continuously.

Observation 4. As $\kappa \to \infty$, $\det[I + \frac{1}{\kappa}G_1(j\omega)G_2(j\omega)] \to 1$.

Observation 5. Suppose that κ is very large such that $\det[I + \frac{1}{\kappa}G_1(j\omega)G_2(j\omega)]$ is almost equal to 1 for all $\omega \in \mathbb{R} \cup \{\pm\infty\}$. Then suppose that κ is continuously decreased towards 1. Suppose that the Nyquist plot of $\det[I + G_1(j\omega)G_2(j\omega)]$ encircles the origin at least once. Then there must exist at least one κ_0 and one ω_0 for which $\det[I + \frac{1}{\kappa_0}G_1(j\omega_0)G_2(j\omega_0)] = 0$.

Thus, a sufficient condition for the Nyquist plot of $\det[I + G_1(j\omega)G_2(j\omega)]$ to make no encirclements of the origin is that, for all $\kappa \in [1, \infty)$ and all $\omega \in \mathbb{R} \cup \{\pm\infty\}$, $\det[I + \frac{1}{\kappa}G_1(j\omega)G_2(j\omega)] \neq 0$. Subsequently, we will present scenarios in which this sufficient condition is satisfied and thus the stability of the negative feedback-loop is guaranteed.

[1]Proof outline: $\det[I + \frac{1}{\kappa}G_1(-j\omega)G_2(-j\omega)] = \det[(I + \frac{1}{\kappa}G_2^*(j\omega)G_1^*(j\omega))^T] = \det[I + \frac{1}{\kappa}G_2^*(j\omega)G_1^*(j\omega)]$ (from [23, Equation 6.1.4]) $= \det[(I + \frac{1}{\kappa}G_1(j\omega)G_2(j\omega))^*] = \overline{\det[I + \frac{1}{\kappa}G_1(j\omega)G_2(j\omega)]}$ (from [23, Exercise 6.1.6]).

3 Simple Feedback-Loop

We first give a rapid proof of a stability result for simple feedback interconnections of systems with "mixed" small gain and passivity properties. A result of this nature appeared in [13]. We utilise the Nyquist discussion presented above. As stated in the introduction, our purposes for doing so are twofold: first, we correct an error in Theorems 1, 6 and 3 of [2, 3] and [14], respectively (in these, the system output signals were assumed to be bounded *a priori*); secondly, the technique paves the way to obtaining new sufficient conditions for the stability of large-scale interconnections of "mixed" systems, which we present in Section 4.

Theorem 6. *Suppose that $M_1 \in \mathcal{RH}_\infty$ and $M_2 \in \mathcal{RH}_\infty$ denote the transfer function matrices of "mixed" subsystems interconnected as depicted in Figure 3 and that this interconnection is well-posed. Suppose that there exist two distinct sets of frequency bands: (a) a set denoted by Ω_p that consists of frequency intervals over which both $M_1(j\omega)$ and $M_2(j\omega)$ have associated with them Property (i) as given in Definition 3; and (b) a set denoted by Ω_s that consists of frequency intervals over which both $M_1(j\omega)$ and $M_2(j\omega)$ have associated with them Property (ii) as given in Definition 3. Furthermore, suppose that $\Omega_p \cup \Omega_s = \mathbb{R} \cup \{\pm\infty\}$. Then the negative feedback-loop is stable.*

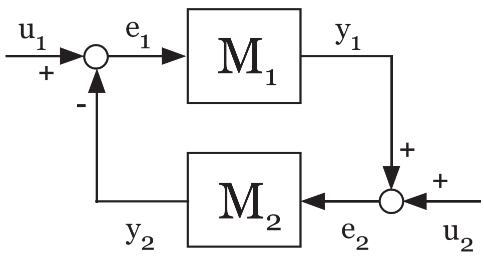

Figure 3: A negative feedback interconnection of "mixed" systems.

Proof. Our aim is to show that, for all $\kappa \in [1, \infty)$ and all $\omega \in \mathbb{R} \cup \{\pm\infty\}$, $\det[I + \frac{1}{\kappa}M_1(j\omega)M_2(j\omega)] \neq 0$. From Section 2, this is a sufficient condition for stability. We do so by splitting our proof into two parts: (i) first, we show that $\det[I + \frac{1}{\kappa}M_1(j\omega)M_2(j\omega)] \neq 0$ for all $\kappa \in [1, \infty)$ and all $\omega \in \Omega_s$; and (ii) then, we show that $\det[I + \frac{1}{\kappa}M_1(j\omega)M_2(j\omega)] \neq 0$ for all $\kappa \in [1, \infty)$ and all $\omega \in \Omega_p$.

Part (i): $\forall \omega \in \Omega_s$**.** From Property (ii) of Definition 3, for $i = 1, 2$, there exists an $\epsilon_i < 1$ such that $-M_i^*(j\omega)M_i(j\omega) + \epsilon_i^2 I \geq 0$. This implies that, for $i = 1, 2$, $\bar{\sigma}(M_i(j\omega)) < 1$ which implies that $\bar{\sigma}(M_1(j\omega)M_2(j\omega)) < 1$ since $\bar{\sigma}(M_1(j\omega)M_2(j\omega)) \leq \bar{\sigma}(M_1(j\omega))\bar{\sigma}(M_2(j\omega))$. Now

$$0 < 1 - \bar{\sigma}(M_1(j\omega)M_2(j\omega)) \leq \underline{\sigma}(I + M_1(j\omega)M_2(j\omega))$$

from [21, Section 2.8] and so $\underline{\sigma}(I + M_1(j\omega)M_2(j\omega)) \neq 0$ which is equivalent to $\det[I + M_1(j\omega)M_2(j\omega)] \neq 0$. Furthermore, $\det[I + \frac{1}{\kappa}M_1(j\omega)M_2(j\omega)] \neq 0$ for any $\kappa > 1$. This is because $\bar{\sigma}(M_1(j\omega)M_2(j\omega)) < 1$ is equivalent to $\frac{1}{\kappa}\bar{\sigma}(M_1(j\omega)M_2(j\omega)) < \frac{1}{\kappa}$ (which is < 1) for any $\kappa > 1$, and so $\bar{\sigma}\left(\frac{1}{\kappa}M_1(j\omega)M_2(j\omega)\right) < 1$ for any $\kappa > 1$. Then

$$0 < 1 - \bar{\sigma}\left(\frac{1}{\kappa}M_1(j\omega)M_2(j\omega)\right) \leq \underline{\sigma}\left(I + \frac{1}{\kappa}M_1(j\omega)M_2(j\omega)\right)$$

for any $\kappa > 1$ and from this the determinant inequality is immediate.

Part (ii): $\forall\omega \in \Omega_p$**.** From Property (i) of Definition 3, for $i = 1, 2$, there exist $k_i, l_i > 0$ such that $-k_i M_i^*(j\omega)M_i(j\omega) + M_i^*(j\omega) + M_i(j\omega) - l_i I \geq 0$. This implies that, for $i = 1, 2$, $M_i^*(j\omega) + M_i(j\omega) > 0$. Observe that $M_i^*(j\omega) + M_i(j\omega) > 0$ if and only if $\frac{1}{\sqrt{\kappa}}M_i^*(j\omega) + \frac{1}{\sqrt{\kappa}}M_i(j\omega) > 0$, where $\kappa > 0$. Then, from Lemma 1, $\det[I + \frac{1}{\kappa}M_1(j\omega)M_2(j\omega)] \neq 0$ for any $\kappa > 0$ and hence for any $\kappa \geq 1$. □

4 Large-Scale Interconnections

Building on the techniques of the previous section, we now present sufficient conditions for the stability of large-scale interconnections of systems with mixtures of small gain and passivity properties. Consider a linear interconnection of N "mixed" systems with square transfer function matrices denoted by $M_i \in \mathcal{RH}_\infty$, $i = 1, \ldots, N$. The interconnection will be described by

$$e_i = u_i - \sum_{j=1}^{N} H_{ij} y_j,$$

where e_i is the input to subsystem i, $y_i = M_i e_i$ is the output of subsystem i, u_i is an external input and H_{ij} is a matrix with real, constant entries. Writing

$$\boldsymbol{e} := \begin{pmatrix} e_1 \\ \vdots \\ e_N \end{pmatrix}, \boldsymbol{y} := \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix} \text{ and } \boldsymbol{u} := \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix},$$

the interconnection description may be written more compactly as

$$\boldsymbol{e} = \boldsymbol{u} - H\boldsymbol{y}, \tag{1}$$

where H is a matrix with block entries H_{ij}. Let $\tilde{M} := \text{diag}(M_1, \ldots, M_N)$ such that $\boldsymbol{y} = \tilde{M}\boldsymbol{e}$. Eliminating $\boldsymbol{y}$ from (1), we have

$$\boldsymbol{e} = (I + H\tilde{M})^{-1}\boldsymbol{u}.$$

Then

$$\boldsymbol{y} = \tilde{M}(I + H\tilde{M})^{-1}\boldsymbol{u}. \tag{2}$$

This set-up is depicted in Figure 4. We will assume that the interconnection is well-posed and, similarly to Theorem 6, impose the following extra conditions on the systems in the interconnection. We require the existence of two distinct sets of frequency bands: (a) a set denoted by Ω_p that consists of frequency intervals over which every $M_i(j\omega)$ has Property (i) as given in Definition 3 associated with it; and (b) a set denoted by Ω_s that consists of frequency intervals over which every $M_i(j\omega)$ has Property (ii) as given in Definition 3 associated with it. Again, we also require that $\Omega_p \cup \Omega_s = \mathbb{R} \cup \{\pm\infty\}$. In the following, $p_i, q_i \in \mathbb{R}$ for $i = 1, \ldots, N$.

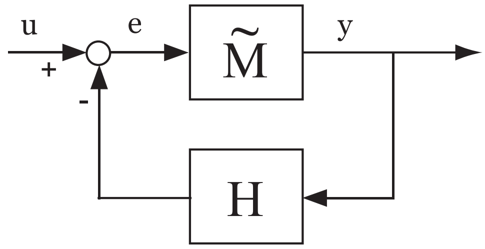

Figure 4: A large-scale interconnection of "mixed" systems.

Theorem 7. *An interconnection of "mixed" subsystems, with input* $\boldsymbol{u}$ *and output* $\boldsymbol{y}$*, as described above, is stable if there exist positive definite matrices* $P := diag(p_1 I, \ldots, p_N I)$ *and* $Q := diag(q_1 I, \ldots, q_N I)$ *such that* $H^T Q + QH > 0$ *and* $-H^T PH + P > 0$.

Proof. Similarly to Section 3, our aim is to show that, for all $\kappa \in [1, \infty)$ and all $\omega \in \mathbb{R} \cup \{\pm\infty\}$, $\det[I + \frac{1}{\kappa} H\tilde{M}(j\omega)] \neq 0$. Again, we split our proof into two parts: (i) first, we show that $\det[I + \frac{1}{\kappa} H\tilde{M}(j\omega)] \neq 0$ for all $\kappa \in [1, \infty)$ and all $\omega \in \Omega_s$; and (ii) then, we show that $\det[I + \frac{1}{\kappa} H\tilde{M}(j\omega)] \neq 0$ for all $\kappa \in [1, \infty)$ and all $\omega \in \Omega_p$.

Part (i): $\forall \omega \in \Omega_s$**.** Suppose that there exists a positive definite matrix P (as defined above) such that $-H^T PH + P > 0$. Let $\tilde{P} := P^{\frac{1}{2}}$ and note that $\tilde{P}^T = \tilde{P}$ [24]. Now $-H^T \tilde{P}^2 H + \tilde{P}^2 = \tilde{P}^T(-\tilde{P}^{-T} H^T \tilde{P}^T \tilde{P} H \tilde{P}^{-1} + I)\tilde{P}$ and so $-(\tilde{P} H \tilde{P}^{-1})^T \tilde{P} H \tilde{P}^{-1} + I > 0$ since $-H^T PH + P$ and $-(\tilde{P} H \tilde{P}^{-1})^T \tilde{P} H \tilde{P}^{-1} + I$ are Hermitian congruent. Set $H_P := \tilde{P} H \tilde{P}^{-1}$. Then $-H_P^T H_P + I > 0$. Equivalently, $\bar{\sigma}(H_P) < 1$. From Property (ii) of Definition 3, for $i = 1, \ldots, N$, there exists an $\epsilon_i < 1$ such that $-M_i^*(j\omega) M_i(j\omega) + \epsilon_i^2 I \geq 0$. This implies that, for $i = 1, \ldots, N$, $-M_i^*(j\omega) M_i(j\omega) + I > 0$. Since $\bar{\sigma}(M_i(j\omega)) < 1$, the same is true for $\tilde{M}(j\omega)$, ie: $\bar{\sigma}(\tilde{M}(j\omega)) < 1$. Then $\bar{\sigma}(H_P \tilde{M}(j\omega)) < 1$. Now

$$0 < 1 - \bar{\sigma}(H_P \tilde{M}(j\omega)) \leq \underline{\sigma}(I + H_P \tilde{M}(j\omega))$$

from [21, Section 2.8] and so $\underline{\sigma}(I + H_P \tilde{M}(j\omega)) \neq 0$ which is equivalent to $\det[I + H_P \tilde{M}(j\omega)] \neq 0$. Furthermore, $\det[I + \frac{1}{\kappa} H_P \tilde{M}(j\omega)] \neq 0$ for any $\kappa > 1$. This is because $\bar{\sigma}(H_P \tilde{M}(j\omega)) < 1$ is equivalent to $\frac{1}{\kappa}\bar{\sigma}(H_P \tilde{M}(j\omega)) < \frac{1}{\kappa}$ (which is < 1) for any $\kappa > 1$, and so $\bar{\sigma}\big(\frac{1}{\kappa} H_P \tilde{M}(j\omega)\big) < 1$ for any $\kappa > 1$. Then

$$0 < 1 - \bar{\sigma}\left(\frac{1}{\kappa} H_P \tilde{M}(j\omega)\right) \leq \underline{\sigma}\left(I + \frac{1}{\kappa} H_P \tilde{M}(j\omega)\right)$$

for any $\kappa > 1$. Finally, note that $\det[I + \frac{1}{\kappa}H_P\tilde{M}(j\omega)] = \det[\tilde{P}]\det[I + \frac{1}{\kappa}H\tilde{M}(j\omega)]\det[\tilde{P}^{-1}]$ since $\tilde{P}^{-1}$ and $\tilde{M}(j\omega)$ commute.

Part (ii): $\forall\omega \in \Omega_p$**.** Suppose that there exists a positive definite matrix Q (as defined above) such that $H^TQ + QH > 0$. Let $\tilde{Q} := Q^{\frac{1}{2}}$ and note that $\tilde{Q}^T = \tilde{Q}$ [24]. Now $H^T\tilde{Q}^2 + \tilde{Q}^2H = \tilde{Q}^T(\tilde{Q}^{-T}H^T\tilde{Q}^T + \tilde{Q}H\tilde{Q}^{-1})\tilde{Q}$ and so $\tilde{Q}^{-T}H^T\tilde{Q}^T + \tilde{Q}H\tilde{Q}^{-1} > 0$ since $H^TQ + QH$ and $\tilde{Q}^{-T}H^T\tilde{Q}^T + \tilde{Q}H\tilde{Q}^{-1}$ are Hermitian congruent. Set $H_Q := \tilde{Q}H\tilde{Q}^{-1}$. Then $H_Q^T + H_Q > 0$. From Property (i) of Definition 3, for $i = 1, \ldots, N$, there exist $k_i, l_i > 0$ such that $-k_iM_i^*(j\omega)M_i(j\omega) + M_i^*(j\omega) + M_i(j\omega) - l_iI \geq 0$. This implies that, for $i = 1, \ldots, N$, $M_i^*(j\omega) + M_i(j\omega) > 0$. Hence, the same is true for $\tilde{M}(j\omega)$, ie: $\tilde{M}^*(j\omega) + \tilde{M}(j\omega) > 0$. Observe that $\tilde{M}^*(j\omega) + \tilde{M}(j\omega) > 0$ if and only if $\frac{1}{\kappa}\tilde{M}^*(j\omega) + \frac{1}{\kappa}\tilde{M}(j\omega) > 0$, where $\kappa > 0$. Then, from Lemma 1, $\det[I + \frac{1}{\kappa}H_Q\tilde{M}(j\omega)] \neq 0$ for any $\kappa > 0$ and hence for any $\kappa \geq 1$. Finally, note that $\det[I + \frac{1}{\kappa}H_Q\tilde{M}(j\omega)] = \det[\tilde{Q}]\det[I + \frac{1}{\kappa}H\tilde{M}(j\omega)]\det[\tilde{Q}^{-1}]$ since $\tilde{Q}^{-1}$ and $\tilde{M}(j\omega)$ commute and so $\det[I + \frac{1}{\kappa}H\tilde{M}(j\omega)] \neq 0$ for any $\kappa \geq 1$. □

Fixing $P = Q = I$ in the above theorem statement gives the following result.

Corollary 8. *An interconnection of "mixed" subsystems, with input* $\boldsymbol{u}$ *and output* $\boldsymbol{y}$*, as described above, is stable if* $H^T + H > 0$ *and* $-H^TH + I > 0$*.*

Our next version of the large-scale interconnected "mixed" systems stability result involves some relaxation on the requirements of the interconnection structure described by the matrix H compared to the conditions on H specified in Theorem 7. This relaxation is achieved by taking into account the values of k_i and ϵ_i associated with each of the "mixed" subsystems in the interconnection, where ϵ_i denotes the gain of the ith "mixed" system over frequencies in Ω_s, while k_i provides a measure of output strict passivity for the ith "mixed" system over frequencies in Ω_p. Suppose that $K := \text{diag}(k_1I, \ldots, k_NI)$ and $E := \text{diag}(\epsilon_1I, \ldots, \epsilon_NI)$, where $k_i > 0$ and $0 < \epsilon_i < 1$ for $i = 1, \ldots, N$.

Theorem 9. *An interconnection of "mixed" subsystems, with input* $\boldsymbol{u}$ *and output* $\boldsymbol{y}$*, as described above, is stable if there exist positive definite matrices* $P := \text{diag}(p_1I, \ldots, p_NI)$ *and* $Q := \text{diag}(q_1I, \ldots, q_NI)$ *such that* $H^TQ + QH + QK > 0$ *and* $-H^TPE^2H + P > 0$*.*

Proof. The proof follows in a manner similar to that of Theorem 7's proof. As before, we want to show that $\det[I + \frac{1}{\kappa}H\tilde{M}(j\omega)] \neq 0$ for all $\kappa \in [1, \infty)$ and all $\omega \in \mathbb{R} \cup \{\pm\infty\}$.

Part (i): $\forall\omega \in \Omega_s$**.** Suppose that there exists a positive definite matrix P (as defined above) such that $-H^TPE^2H + P > 0$. Then, similarly to the proof of Theorem 7, we obtain $\bar{\sigma}(EH_P) < 1$, where $H_P := P^{\frac{1}{2}}H(P^{\frac{1}{2}})^{-1}$. From Property (ii) of Definition 3, for $i = 1, \ldots, N$, there exists an $\epsilon_i < 1$ such that $-M_i^*(j\omega)M_i(j\omega) + \epsilon_i^2I \geq 0$. Equivalently, for $i = 1, \ldots, N$, there exists an $\epsilon_i < 1$ such that $-\frac{1}{\epsilon_i^2}M_i^*(j\omega)M_i(j\omega) + I \geq 0$. Since $\bar{\sigma}(\frac{1}{\epsilon_i}M_i(j\omega)) \leq 1$, $\bar{\sigma}(E^{-1}\tilde{M}(j\omega)) \leq 1$. Then $\bar{\sigma}(\tilde{M}(j\omega)H_P) < 1$ since E^{-1} and $\tilde{M}(j\omega)$ commute. Now

$$0 < 1 - \bar{\sigma}(\tilde{M}(j\omega)H_P) \leq \underline{\sigma}(I + \tilde{M}(j\omega)H_P)$$

from [21, Section 2.8] and so $\underline{\sigma}(I+\tilde{M}(j\omega)H_P) \neq 0$ which is equivalent to $\det[I+\tilde{M}(j\omega)H_P] \neq 0$. Furthermore, $\det[I + \frac{1}{\kappa}\tilde{M}(j\omega)H_P] \neq 0$ for any $\kappa > 1$. This is because $\bar{\sigma}(\tilde{M}(j\omega)H_P) < 1$ is equivalent to $\frac{1}{\kappa}\bar{\sigma}(\tilde{M}(j\omega)H_P) < \frac{1}{\kappa}$ (which is < 1) for any $\kappa > 1$, and so $\bar{\sigma}\big(\frac{1}{\kappa}\tilde{M}(j\omega)H_P\big) < 1$ for any $\kappa > 1$. Then

$$0 < 1 - \bar{\sigma}\left(\frac{1}{\kappa}\tilde{M}(j\omega)H_P\right) \leq \underline{\sigma}\left(I + \frac{1}{\kappa}\tilde{M}(j\omega)H_P\right)$$

for any $\kappa > 1$. Finally, note that $\det[I + \frac{1}{\kappa}\tilde{M}(j\omega)H_P] = \det[I + \frac{1}{\kappa}H_P\tilde{M}(j\omega)]$ for any $\kappa \geq 1$ [25, page 651] [23, Exercise 6.2.7] and that $\det[I + \frac{1}{\kappa}H_P\tilde{M}(j\omega)] = \det[P^{\frac{1}{2}}]\det[I + \frac{1}{\kappa}H\tilde{M}(j\omega)]\det[(P^{\frac{1}{2}})^{-1}]$ since $(P^{\frac{1}{2}})^{-1}$ and $\tilde{M}(j\omega)$ commute.

Part (ii): $\forall\omega \in \Omega_p$. Suppose that there exists a positive definite matrix Q (as defined above) such that $H^TQ + QH + QK > 0$. Similarly to the proof of Theorem 7, we obtain $H_Q^T + H_Q + K > 0$, where $H_Q := Q^{\frac{1}{2}}H(Q^{\frac{1}{2}})^{-1}$. From Property (i) of Definition 3, for $i = 1,\ldots,N$, there exist $k_i, l_i > 0$ such that $-k_iM_i^*(j\omega)M_i(j\omega)+M_i^*(j\omega)+M_i(j\omega)-l_iI \geq 0$. This implies that, for $i = 1,\ldots,N$, $-k_iM_i^*(j\omega)M_i(j\omega) + M_i^*(j\omega) + M_i(j\omega) > 0$. Hence, $-\tilde{M}^*(j\omega)K\tilde{M}(j\omega) + \tilde{M}^*(j\omega) + \tilde{M}(j\omega) > 0$. Then, from Lemma 4, $\det[I + \frac{1}{\kappa}\tilde{M}(j\omega)H_Q] \neq 0$ for any $\kappa \geq 1$. Finally, note that $\det[I+\frac{1}{\kappa}\tilde{M}(j\omega)H_Q] = \det[I+\frac{1}{\kappa}H_Q\tilde{M}(j\omega)]$ [25, page 651] [23, Exercise 6.2.7], and that $\det[I + \frac{1}{\kappa}H_Q\tilde{M}(j\omega)] = \det[Q^{\frac{1}{2}}]\det[I + \frac{1}{\kappa}H\tilde{M}(j\omega)]\det[(Q^{\frac{1}{2}})^{-1}]$ since $(Q^{\frac{1}{2}})^{-1}$ and $\tilde{M}(j\omega)$ commute. □

Set $P = Q = I$ in the above theorem statement to obtain the following corollary.

Corollary 10. *An interconnection of "mixed" subsystems, with input* $\boldsymbol{u}$ *and output* $\boldsymbol{y}$*, as described above, is stable if* $H^T + H + K > 0$ *and* $-H^TE^2H + I > 0$.

We now compare our large-scale interconnected "mixed" systems stability results to the large-scale interconnected systems stability results of [1, Sections IV and V] (eg: see [1, Theorems 4 and 5]). In [1], a sufficient condition for the stability of large-scale interconnections of passive systems is the existence of a positive definite diagonal matrix Q such that $H^TQ + QH > 0$. A necessary condition for this linear matrix inequality (LMI) to be feasible is that H has all eigenvalues with positive real parts [16, Theorem 1 of Section 13.1]. Similarly in [1], a sufficient condition for the stability of large-scale interconnections of systems with finite gain is the existence of a positive definite diagonal matrix P such that $-H^TPE^2H + P > 0$.[2] A necessary condition for this LMI to be feasible is that all of the eigenvalues of EH lie inside the unit circle centred at the origin of the complex plane [26, Theorem 5.18]. Our results show that, as one relaxes the assumptions on the subsystems in an interconnection from passivity or finite gain to "mixedness" the [1]-like conditions on the interconnection matrix become more stringent, ie: more restriction is imposed on the structure of the interconnection, ie: the matrix H has to "work harder" in order for stability to be guaranteed. For instance, in Theorems 7 and 9, the existence of solutions to a pair of

[2]Note that, in [1], the gains ϵ_i appearing in E are not necessarily less than one.

LMIs, as opposed to a single LMI, is sufficient for stability; we illustrate this point further with the following example.

Example 1. Consider the example of an interconnected system from [1], depicted in Figure 5, with interconnection matrix

$$H = \begin{bmatrix} 1 & 0 & -\gamma \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

Assume that G_1, G_2 and G_3 are passive and that $-8 < \gamma < 1$. According to [1], under these conditions, one should be able to find a positive definite diagonal matrix Q such that $H^TQ + QH > 0$ which thus means that the interconnected system is stable. Using the Robust Control Toolbox (MATLAB R2009a) we verify that, for any $-8 < \gamma < 1$, finding a solution to the LMI $H^TQ + QH > 0$ is indeed feasible.

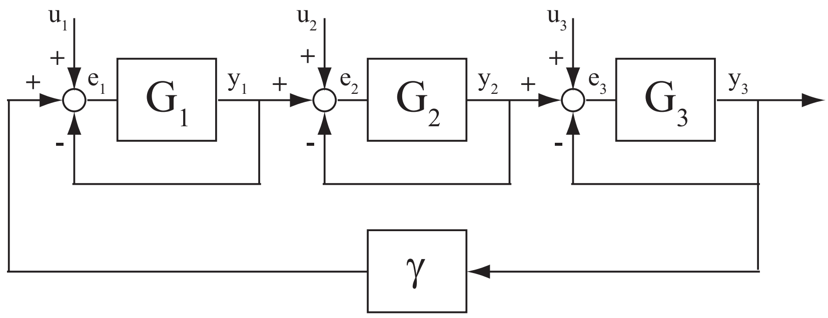

Figure 5: Example 1.

Now, suppose that we relax the conditions on G_1, G_2 and G_3 and assume that they are all “mixed” systems. For the same values of γ, we search for positive definite diagonal matrices P and Q that satisfy $H^TQ + QH > 0$ and $-H^TPH + P > 0$ simultaneously. We find that this LMI problem is not feasible for any $-8 < \gamma < 1$. ■

We conclude this section with an example of an interconnection of “mixed” systems for which stability is guaranteed.

Example 2. Consider the interconnection of systems depicted in Figure 6 and suppose that M_1, M_2 and M_3 are “mixed” with $k_1 = k_2 = k_3 = 0.01$. Let $\gamma = 0.5$. Then $K = 0.01I$ and

$$H = \begin{bmatrix} 0 & 0 & 0.5 \\ 0 & 0 & 0.5 \\ -0.5 & -0.5 & 0 \end{bmatrix}.$$

Since the eigenvalues of $H^T + H + K$ and $I - H^TH$ are positive, the interconnection is stable by Corollaries 8 and 10. ■

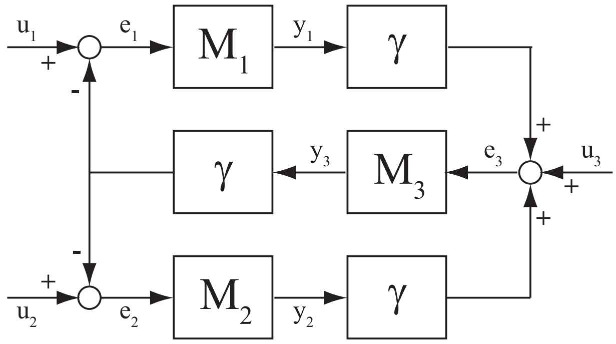

Figure 6: Example 2.

5 Time-Varying Interconnections of SPR Systems

The final contribution of this paper concerns obtaining a stability result for large-scale, time-varying interconnections of SISO, strictly proper, SPR systems. This stability condition mirrors the condition obtained in [1] for time-invariant interconnections of passive systems and is thus an extension of this work.

Consider the N SISO, LTI systems

$$\begin{aligned} \dot{x}_i &= A_i x_i + b_i e_i, \\ y_i &= c_i^T x_i, \end{aligned}$$

$i = 1, \ldots, N$, where $x_i(t) \in \mathbb{R}^{n_i \times 1}$, $A_i \in \mathbb{R}^{n_i \times n_i}$, $b_i \in \mathbb{R}^{n_i \times 1}$, $c_i \in \mathbb{R}^{n_i \times 1}$ and A_i is Hurwitz, with transfer functions $G_i(s) := c_i^T (sI - A_i)^{-1} b_i$. Suppose that (A_i, b_i) is controllable and (c_i^T, A_i) is observable for $i = 1, \ldots, N$. Define the vectors

$$\boldsymbol{x} := \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}, \ \boldsymbol{e} := \begin{bmatrix} e_1 \\ \vdots \\ e_N \end{bmatrix} \text{ and } \boldsymbol{y} := \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}$$

and let

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{e}, \tag{3}$$

$$\boldsymbol{y} = C^T \boldsymbol{x}, \tag{4}$$

where $A := \mathrm{diag}(A_1, \ldots, A_N) \in \mathbb{R}^{(n_1 + \cdots + n_N) \times (n_1 + \cdots + n_N)}$, $B := \mathrm{diag}(b_1, \ldots, b_N) \in \mathbb{R}^{(n_1 + \cdots + n_N) \times N}$ and $C := \mathrm{diag}(c_1, \ldots, c_N) \in \mathbb{R}^{(n_1 + \cdots + n_N) \times N}$. Then (A, B) is controllable, (C^T, A) is observable and A is Hurwitz. Denote the transfer function of this new system as $G(s) := C^T (sI - A)^{-1} B = \mathrm{diag}(G_1(s), \ldots, G_N(s))$.

Let $H(t)$ be some matrix with real entries that are bounded, continuous functions of time, that describes how the N subsystems are interconnected at time $t \geq 0$, as follows:

$$\boldsymbol{e} = -H(t)\boldsymbol{y}. \tag{5}$$

Substituting (5) and (4) into (3) gives

$$\dot{\boldsymbol{x}} = [A - BH(t)C^T]\boldsymbol{x}. \tag{6}$$

Now, suppose that $G(s)$ is SPR [20, Section 2.14], [27, Definition 8.5], [28, Definition 5.18]. Then $QG(s)$ is SPR for any positive definite matrix $Q := \mathrm{diag}(q_1, \ldots, q_N)$, where $q_i \in \mathbb{R}$ for $i = 1, \ldots, N$, and the KYP lemma [20, Section 3.1.4], [27, Lemma 8.1], [28, Theorem 5.14] states that there exists a positive definite matrix $P \in \mathbb{R}^{(n_1+\cdots+n_N)\times(n_1+\cdots+n_N)}$ such that

$$\begin{aligned} A^T P + PA &< 0, \\ PB &= CQ. \end{aligned}$$

Define $V(x) = x^T P x$ as a candidate Lyapunov function for (6). Then

$$\begin{aligned} \dot{V}(x,t) &= \dot{x}^T P x + x^T P \dot{x} \\ &= x^T [A - BH(t)C^T]^T P x + x^T P [A - BH(t)C^T] x \\ &= x^T [A^T P + PA - PBH(t)C^T - CH(t)^T (PB)^T] x \\ &= x^T [A^T P + PA - CQH(t)C^T - CH(t)^T QC^T] x \\ &= x^T [A^T P + PA] x - x^T C [H^T(t) Q + QH(t)] C^T x. \end{aligned}$$

This derivative function is negative definite if $H^T(t)Q + QH(t) \geq 0$ for all $t \geq 0$. Hence, we have the following result.

Theorem 11. *The system described by* (6) *is uniformly asymptotically stable*[3] *if there exists a positive definite matrix* $Q := \mathrm{diag}(q_1, \ldots, q_N)$*, where* $q_i \in \mathbb{R}$ *for* $i = 1, \ldots, N$*, such that* $H^T(t)Q + QH(t) \geq 0$ *for all* $t \geq 0$*.*

While our extension of [1, Theorem 4] to time-varying interconnections of SPR systems is straightforward, further extensions to time-varying interconnections of "mixed" systems do not seem immediate. One reason for this difficulty might relate to the more complex nature of state-space characterisations of "mixed" systems. Generalised KYP lemmas that characterise systems with positive real or bounded real properties over finite frequency bands were derived in [31, 32].

6 Conclusions

The key contributions of this paper concern the derivation of sufficient conditions for: (i) the stability of large-scale, time-invariant interconnections of "mixed" systems; and (ii) the stability of large-scale, time-varying interconnections of SISO, strictly proper, SPR systems. Concerning the first contribution, we showed that relaxing the assumptions on the systems in a large-scale interconnection, from suppositions of passivity or small gain, to assumptions

[3]For a definition of uniform asymptotic stability, see [29, Chapter 5] or [30, Theorem 2.5].

of “mixedness,” results in the [1]-like conditions for stability on the interconnection structure itself becoming more stringent. Such a result has the potential to steer strategies for large-scale system design and is a direction for future research that the authors would like to pursue. In regards to the second contribution, the inspiration for studying time-varying interconnections emerges from applications concerning (for example) mobile vehicle networks, where agents, or vehicles, come in and out of range with each other (ie: links between the systems are created or broken over time). The extension of our result to time-varying interconnections of MIMO, proper, SPR systems seems straightforward and will be published at a later date.

References

[1] P.J. Moylan and D.J. Hill, Stability criteria for large-scale systems, *IEEE Transactions on Automatic Control*, vol. 23, no. 2, 1978, pp. 143-149.

[2] W.M. Griggs, B.D.O. Anderson and A. Lanzon, A “mixed” small gain and passivity theorem for an interconnection of linear time-invariant systems, in *Proceedings of the European Control Conference 2007*, Kos, Greece, 2007, pp. 2410-2416.

[3] W.M. Griggs, B.D.O. Anderson and A. Lanzon, A “mixed” small gain and passivity theorem in the frequency domain, *Systems & Control Letters*, vol. 56, no. 9-10, 2007, pp. 596-602.

[4] C.E. Rohrs, L. Valavani, M. Athans and G. Stein, Robustness of continuous-time adaptive control algorithms in the presence of unmodeled dynamics, *IEEE Transactions on Automatic Control*, vol. 30, no. 9, 1985, pp. 881-889.

[5] B.D.O. Anderson, R.R. Bitmead, C.R. Johnson Jr., P.V. Kokotovic, R.L. Kosut, I.M.Y. Mareels, L. Praly, B.D. Riedle, *Stability of Adaptive Systems: Passivity and Averaging Analysis*, The MIT Press, Cambridge, MA; 1986.

[6] I.M.Y. Mareels, B.D.O. Anderson, R.R. Bitmead, M. Bodson and S.S. Sastry, Revisiting the MIT rule for adaptive control, in *Proceedings of the 2nd IFAC Workshop on Adaptive Systems in Control and Signal Processing*, Lund, Sweden, 1986, pp. 161-166.

[7] B.D.O. Anderson, Failures of adaptive control theory and their resolution, *Communications in Information and Systems*, vol. 5, no. 1, 2005, pp. 1-20.

[8] W.M. Griggs, B.D.O. Anderson and R.N. Shorten, A test for determining systems with “mixed” small gain and passivity properties, *Systems & Control Letters*, vol. 60, no. 7, 2011, pp. 479-485.

[9] I. Lestas and G. Vinnicombe, Scalable decentralized robust stability certificates for networks of interconnected heterogeneous dynamical systems, *IEEE Transactions on Automatic Control*, vol. 51, no. 10, 2006, pp. 1613-1625.

[10] I. Lestas and G. Vinnicombe, Heterogeneity and scalability in group agreement protocols: beyond small gain and passivity approaches, *Automatica*, vol. 46, no. 7, 2010, pp. 1141-1151.

[11] C-Y. Kao, U. Jönsson and H. Fujioka, Characterization of robust stability of a class of interconnected systems, *Automatica*, vol. 45, no. 1, 2009, pp. 217-224.

[12] I. Postlethwaite, J.M. Edmunds and A.G.J. MacFarlane, Principal gains and principal phases in the analysis of linear multivariable feedback systems, *IEEE Transactions on Automatic Control*, vol. 26, no. 1, 1981, pp. 32-46.

[13] J. Bao, P.L. Lee, F. Wang and W. Zhou, New Robust Stability Criterion and Robust Control Synthesis, *International Journal of Robust and Nonlinear Control*, vol. 8, no. 1, 1998, pp. 49-59.

[14] W.M. Griggs, *Stability Results for Feedback Control Systems*, Ph.D. Thesis, The Australian National University, Australia, December 2007.

[15] W.M. Griggs, B.D.O. Anderson and R.N. Shorten, Determining "mixedness" and an application of finite-gain stability results to "mixed" system interconnections, in *Proceedings of the 49th IEEE Conference on Decision and Control*, Atlanta, GA, USA, 2010, pp. 714-719.

[16] P. Lancaster and M. Tismenetsky, *The Theory of Matrices with Applications*, Academic Press, San Diego and London; 1985.

[17] B. Noble and J.W. Daniel, *Applied Linear Algebra*, Prentice Hall, Englewood Cliffs, NJ; 1988.

[18] M.C. Pease, *Methods of Matrix Algebra*, Academic Press, New York and London; 1965.

[19] C.A. Desoer and M. Vidyasagar, *Feedback Systems: Input-Output Properties*, Academic Press, New York and London; 1975.

[20] B. Brogliato, R. Lozano, B. Maschke and O. Egeland, *Dissipative Systems Analysis and Control: Theory and Applications*, Springer, London; 2007.

[21] K. Zhou with J.C. Doyle and K. Glover, *Robust and Optimal Control*, Prentice Hall, Upper Saddle River, NJ; 1996.

[22] S. Skogestad and I. Postlethwaite, *Multivariable Feedback Control: Analysis and Design*, John Wiley & Sons, Chichester; 1996.

[23] C.D. Meyer, *Matrix Analysis and Applied Linear Algebra*, SIAM, Philadelphia, PA; 2000.

[24] A. Wouk, A note on square roots of positive operators, *SIAM Review*, vol. 8, no. 1, 1966, pp. 100-102.

[25] T. Kailath, *Linear Systems*, Prentice Hall, Upper Saddle River, NJ; 1980.

[26] S. Barnett, *Introduction to Mathematical Control Theory*, Oxford University Press, London; 1975.

[27] H.J. Marquez, *Nonlinear Control Systems: Analysis and Design*, John Wiley & Sons, Hoboken, NJ; 2003.

[28] W.M. Haddad and V. Chellaboina, *Nonlinear Dynamical Systems and Control: A Lyapunov-Based Approach*, Princeton University Press, Princeton and Woodstock; 2008.

[29] S. Sastry, *Nonlinear Systems: Analysis, Stability, and Control*, Springer-Verlag, New York, NY; 1999.

[30] Z. Sun and S.S. Ge, *Switched Linear Systems: Control and Design*, Springer-Verlag, London; 2005.

[31] T. Iwasaki, S. Hara and H. Yamauchi, Dynamical system design from a control perspective: finite frequency positive-realness approach, *IEEE Transactions on Automatic Control*, vol. 48, no. 8, 2003, pp. 1337-1354.

[32] T. Iwasaki and S. Hara, Generalized KYP lemma: unified frequency domain inequalities with design applications, *IEEE Transactions on Automatic Control*, vol. 50, no. 1, 2005, pp. 41-59.