A Test for Determining Systems with "Mixed" Small Gain and Passivity Properties*

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Abstract

"Mixedness" is a property that captures elements of the notions of passivity and small gain. In the frequency domain, a linear, time-invariant system is called "mixed" if, over some frequency bands, it is strictly passive and, over the remaining frequencies, it has a gain of less than one; there exist no frequencies over which the system has neither of the notions of these properties associated with it. In this paper, a test is developed for determining whether or not a linear, time-invariant system is "mixed."

1 Introduction

Two important results in the input-output stability theory literature are the small gain and the passivity theorems. The small gain theorem states that if the product of the gains of two stable systems, interconnected via a negative feedback loop, is less than one then the interconnection is stable [1–4]. The passivity theorem guarantees stability of the negative feedback interconnection if, for instance, both of the systems are passive and one of them is input strictly passive with finite gain [1–3, 5]. Circumstances in which the small gain or passivity properties fail to adequately describe a system in question suggest that alternative assumptions may need to be placed on the systems in the interconnection such that stability might be determined.

High frequency dynamics which might destroy the passivity properties of an otherwise passive system lead us to an example of the notion of a "mixed" system provided that, at those destructive high frequency dynamics, the system has a small gain. More generally, in the linear, time-invariant (LTI) case, a system is called "mixed" if, over some frequency

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bands, it has the property of being input and output strictly "passive" and, over the remaining frequencies, it has a "gain" of less than one; there exist no frequencies over which the system has neither of these property notions associated with it. "Mixed" systems were initially discussed in [6,7] (and in the nonlinear case in the time domain in [8,9]) in a dissipative systems' framework [10, 11].

The objective of this paper is to present a necessary and sufficient test for determining whether or not a multi-input, multi-output (MIMO) LTI system is "mixed." The procedure involves the computation of two Hamiltonian matrices, one associated with any potentially passive aspects of the system and the other associated with the notion of system small gain. The examination of the spectral characteristics of these Hamiltonian matrices, which are constructed from state-space data, leads to the elimination of an element of frequency-dependency from the test. The purely imaginary eigenvalues of the Hamiltonian matrices correspond exactly to the frequencies at which zero eigenvalues of certain transfer function matrices typically associated with system passivity and system small gain occur. Testing the sign definiteness of these transfer function matrices at a single frequency point on either side of the frequencies which give rise to the zero eigenvalues yields whether or not the system is "mixed." Spectral conditions for positive realness of transfer function matrices are discussed in [12] and, for more general frequency domain inequalities, in [13].

The paper is divided into the following sections. The notion of a "mixed" system is defined in Section 2. In Section 3, system state-space descriptions are utilised to compute two Hamiltonian matrices and derive associated results which are required for the "mixedness" test described in Section 4. Examples are provided in Section 5.

1.1 Notation

The results of this paper are concerned with LTI systems viewed in the frequency domain. $\mathcal{R}$ denotes the set of proper real rational transfer function matrices. For a transfer function matrix $G(s) \in \mathcal{R}$, $G^{\sim}(s)$ is defined to mean $G^T(-s)$ and $G^*(j\omega) := G^{\sim}(j\omega)$. $\mathcal{L}_\infty$ is a Banach space of matrix- (or scalar-) valued functions that are essentially bounded on $j\mathbb{R}$. The Hardy space, $\mathcal{H}_\infty$, is the closed subspace of $\mathcal{L}_\infty$ with functions that are analytic and bounded in the open right-half plane. In other words, $\mathcal{H}_\infty$ is the space of transfer functions of stable, LTI, continuous-time systems. $\mathcal{RH}_\infty$ denotes the subspace of $\mathcal{H}_\infty$ whose transfer function matrices are proper and real rational. The notation $A \in \mathcal{RH}_\infty^{m\times n}$ will be used to indicate such matrices with m rows and n columns.

2 Definitions and Problem Description

Consider a causal system with square transfer function matrix $M \in \mathcal{RH}_\infty^{m\times m}$. Consider a closed frequency interval $[a, b]$, where $a, b \in \mathbb{R}$.

Definition 1. A causal system with transfer function matrix $M \in \mathcal{RH}_\infty^{m\times m}$ is said to be input and output strictly passive over the frequency interval $[a,b]$ if there exist $k,l>0$ such that

$$-kM^*(j\omega)M(j\omega)+M^*(j\omega)+M(j\omega)-lI\geq 0$$

for all $\omega\in[a,b]$.

Additionally, we will say that the system is input strictly passive over the frequency interval $[a,b]$ if Definition 1 is satisfied with $k=0$; output strictly passive over the frequency interval $[a,b]$ if Definition 1 is satisfied with $l=0$; and passive over the frequency interval $[a,b]$ if Definition 1 is satisfied with $k=l=0$. Definition 1 requires a and b to be finite. In Definition 2, this requirement is relaxed.

Definition 2. Suppose that $\lim_{\omega\to\pm\infty}\det(M^*(j\omega)+M(j\omega))\neq 0$.[1] A causal system with transfer function matrix $M\in\mathcal{RH}_\infty^{m\times m}$ is said to be input and output strictly passive over the frequency interval $(-\infty,a]$, $[b,\infty)$ or $(-\infty,\infty)$ if there exist $k,l>0$ such that

$$-kM^*(j\omega)M(j\omega)+M^*(j\omega)+M(j\omega)-lI\geq 0$$

for all $\omega\in(-\infty,a]$, $[b,\infty)$ or $(-\infty,\infty)$, respectively.

Definition 3. Define the system gain over the frequency interval $[a,b]$ as

$$\epsilon:=\min\{\bar{\epsilon}\in\mathbb{R}_+ : -M^*(j\omega)M(j\omega)+\bar{\epsilon}^2I\geq 0 \text{ for all } \omega\in[a,b]\}.$$

The system is said to have a gain of less than one over the frequency interval $[a,b]$ if $\epsilon<1$.

Definition 4. Suppose that $\lim_{\omega\to\pm\infty}\det(-M^*(j\omega)M(j\omega)+I)\neq 0$.[2] Define the system gain over the frequency interval $(-\infty,a]$, $[b,\infty)$ or $(-\infty,\infty)$ as

$$\epsilon:=\inf\{\bar{\epsilon}\in\mathbb{R}_+ : -M^*(j\omega)M(j\omega)+\bar{\epsilon}^2I\geq 0 \text{ for all } \omega\in(-\infty,a],\ [b,\infty) \text{ or } (-\infty,\infty), \text{ respectively}\}.$$

The system is said to have a gain of less than one over the frequency interval $(-\infty,a]$, $[b,\infty)$ or $(-\infty,\infty)$, respectively, if $\epsilon<1$.

We now define a "mixed" system.

Definition 5. A causal system with transfer function matrix $M\in\mathcal{RH}_\infty^{m\times m}$ is said to be "mixed" if, for each frequency $\omega\in\mathbb{R}$: either (i) $-kM^*(j\omega)M(j\omega)+M^*(j\omega)+M(j\omega)-lI\geq 0$; and/or (ii) $-M^*(j\omega)M(j\omega)+\epsilon^2I\geq 0$. The constants $k,l>0$ and $\epsilon<1$ are independent of ω.

[1] More specifically, suppose that $\lim_{\omega\to\pm\infty}\det(M^*(j\omega)+M(j\omega))=c_p>0$, where $c_p\in\mathbb{R}$.

[2] More specifically, suppose that $\lim_{\omega\to\pm\infty}\det(-M^*(j\omega)M(j\omega)+I)=c_s>0$, where $c_s\in\mathbb{R}$.

Examples of "mixed" systems are systems with the transfer functions

$$m_1(s) = \frac{3s+2}{s+5}, \; m_2(s) = \frac{2s-1}{2s+4}$$

and

$$m_3(s) = \frac{3}{(s+1)(s+2)}.$$

The systems described by the transfer functions

$$m_4(s) = \frac{2s-3}{s+4}$$

and

$$m_5(s) = \frac{10}{(s+1)(s+2)}$$

are not "mixed." To illustrate, consider the Nyquist diagrams of $m_2(s)$ and $m_5(s)$ as shown in Figures 1 and 2 below. From Figure 1, it is evident that there exists some frequency Ω such that the system described by the transfer function $m_2(s)$ is input and output strictly passive over the frequency ranges $(-\infty, -\Omega]$ and $[\Omega, \infty)$ and has a gain of less than one over the frequency range $[-\Omega, \Omega]$. For instance, one could take $\Omega = 3$. This is not the case for the system described by the transfer function $m_5(s)$.

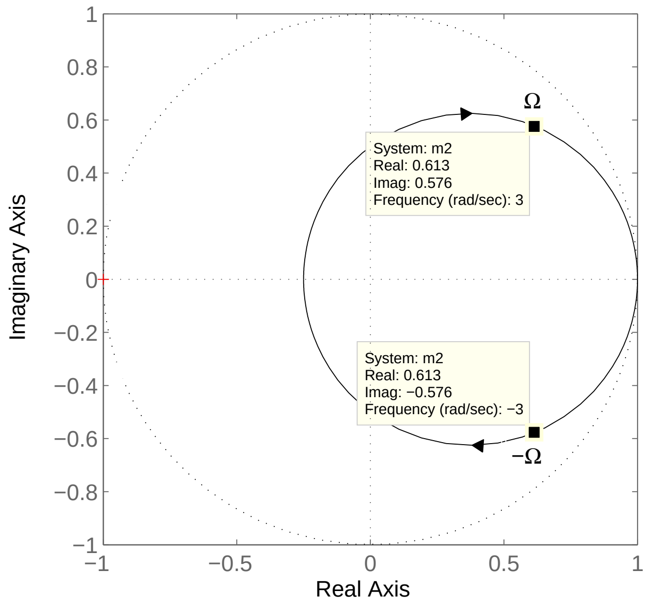

Figure 1: Nyquist diagram of $m_2(s)$.

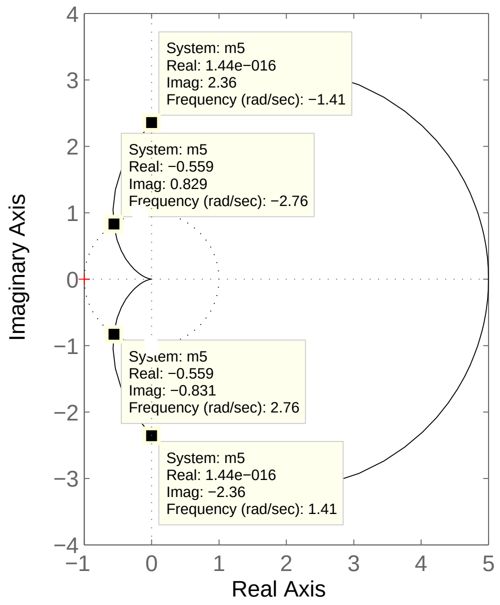

Figure 2: Nyquist diagram of $m_5(s)$.

Remark 1. The requirement that a system is input and output strictly passive over all frequencies $\omega \in \mathbb{R}$ is typically a severe restriction, rarely satisfied by physical systems [2, Section 8.6]. No strictly proper system satisfies this requirement. However, a strictly proper system can be "mixed."

Remark 2. We note the following:

- there exist $k, l > 0$ such that $-kM^*(j\omega)M(j\omega) + M^*(j\omega) + M(j\omega) - lI \geq 0$ for all $\omega \in [a, b]$ if and only if $M^*(j\omega) + M(j\omega) > 0$ for all $\omega \in [a, b]$;

- under the assumption that $\lim_{\omega \to \pm\infty} \det(M^*(j\omega) + M(j\omega)) \neq 0$, there exist $k, l > 0$ such that $-kM^*(j\omega)M(j\omega) + M^*(j\omega) + M(j\omega) - lI \geq 0$ for all $\omega \in (-\infty, a]$, $[b, \infty)$ or $(-\infty, \infty)$ if and only if $M^*(j\omega) + M(j\omega) > 0$ for all $\omega \in (-\infty, a]$, $[b, \infty)$ or $(-\infty, \infty)$, respectively;

- there exists $\epsilon < 1$ such that $-M^*(j\omega)M(j\omega) + \epsilon^2 I \geq 0$ for all $\omega \in [a, b]$ if and only if $-M^*(j\omega)M(j\omega) + I > 0$ for all $\omega \in [a, b]$;

- under the assumption that $\lim_{\omega \to \pm\infty} \det(-M^*(j\omega)M(j\omega) + I) \neq 0$, there exists $\epsilon < 1$ such that $-M^*(j\omega)M(j\omega) + \epsilon^2 I \geq 0$ for all $\omega \in (-\infty, a]$, $[b, \infty)$ or $(-\infty, \infty)$ if and only if $-M^*(j\omega)M(j\omega) + I > 0$ for all $\omega \in (-\infty, a]$, $[b, \infty)$ or $(-\infty, \infty)$, respectively.

Then, for example, provided that $\lim_{\omega\to\pm\infty}\det(M^*(j\omega)+M(j\omega))\neq 0$ and $\lim_{\omega\to\pm\infty}\det(-M^*(j\omega)M(j\omega)+I)\neq 0$, the following holds. The proof follows directly from Remark 2 and Definition 5.

Lemma 1. *Suppose that* $\lim_{\omega\to\pm\infty}\det(M^*(j\omega)+M(j\omega))\neq 0$ *and* $\lim_{\omega\to\pm\infty}\det(-M^*(j\omega)M(j\omega)+I)\neq 0$. *Then a causal system with transfer function matrix* $M\in\mathcal{RH}_\infty^{m\times m}$ *is "mixed" if and only if, at each frequency* $\omega\in\mathbb{R}$*, either* $M^*(j\omega)+M(j\omega)>0$ *and/or* $-M^*(j\omega)M(j\omega)+I>0$.

We now construct a test for determining whether or not a given system is "mixed." For single-input, single-output (SISO) LTI systems, the development of such a test is in one sense redundant as one can, for example, examine the properties of the candidate system graphically via its Nyquist plot. However, analytic tests of "mixedness" for MIMO, LTI systems (and eventually, tests for nonlinear systems) are potentially more useful. In this paper, we explore the MIMO, LTI case.

3 State-Space Descriptions and Hamiltonian Matrices

Suppose that one is given a causal system with stable, square transfer function matrix $M=C(sI-A)^{-1}B+D$ which is described by the equations

$$\begin{aligned}\dot{x} &= Ax+Be,\ \ x(t_0)=x_0,\\ y &= Cx+De,\end{aligned}$$

where $x(t)\in\mathbb{R}^n$, $e(t)\in\mathbb{R}^m$, $y(t)\in\mathbb{R}^m$ and $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times m}$, $C\in\mathbb{R}^{m\times n}$, $D\in\mathbb{R}^{m\times m}$ with A Hurwitz.

From Definitions 1 and 2, $-kM^*(j\omega)M(j\omega)+M^*(j\omega)+M(j\omega)-lI\geq 0$ can be written as

$$\begin{aligned}&-k(B^*(j\omega I-A)^{-*}C^*+D^*)(C(j\omega I-A)^{-1}B+D)+B^*(j\omega I-A)^{-*}C^*+D^*\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad+C(j\omega I-A)^{-1}B+D-lI\geq 0.\end{aligned}$$

Noting that $(j\omega)^*=-j\omega$ gives

$$\begin{aligned}&-k(-B^T(j\omega I+A^T)^{-1}C^T+D^T)(C(j\omega I-A)^{-1}B+D)-B^T(j\omega I+A^T)^{-1}C^T+D^T\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad+C(j\omega I-A)^{-1}B+D-lI\geq 0.\end{aligned}$$

A final rearrangement gives $G_1(j\omega)\geq 0$, where

$$\begin{aligned}G_1(j\omega) :=&\begin{pmatrix}(I-kD)^TC & -B^T\end{pmatrix}\left[j\omega I-\begin{pmatrix}A & 0\\ -kC^TC & -A^T\end{pmatrix}\right]^{-1}\begin{pmatrix}B\\ C^T(I-kD)\end{pmatrix}\\ &-kD^TD+D^T+D-lI.\end{aligned}$$

Similarly, $-M^*(j\omega)M(j\omega)+\epsilon^2 I \geq 0$ from Definitions 3 and 4 may be written as

$$-(B^*(j\omega I-A)^{-*}C^*+D^*)(C(j\omega I-A)^{-1}B+D)+\epsilon^2 I \geq 0.$$

Again, noting that $(j\omega)^*=-j\omega$ gives

$$-(-B^T(j\omega I+A^T)^{-1}C^T+D^T)(C(j\omega I-A)^{-1}B+D)+\epsilon^2 I \geq 0$$

and a final rearrangement gives $G_2(j\omega)\geq 0$, where

$$G_2(j\omega) := \begin{pmatrix} -D^TC & -B^T \end{pmatrix} \left[j\omega I - \begin{pmatrix} A & 0 \\ -C^TC & -A^T \end{pmatrix} \right]^{-1} \begin{pmatrix} B \\ -C^TD \end{pmatrix} - D^TD + \epsilon^2 I.$$

The matrices $G_1(j\omega)$ and $G_2(j\omega)$ are Hermitian matrices and so have real eigenvalues. Note the following two results.

Lemma 2. *Suppose $k,l \in \mathbb{R}$. The matrix $-kM^*(j\omega)M(j\omega)+M^*(j\omega)+M(j\omega)-lI$ has no zero eigenvalues over $\omega \in [a,b]$ if and only if H_1 does not have any eigenvalues on the imaginary axis between and including $-ja$ and $-jb$, where*

$$H_1 := \begin{pmatrix} -A+BX_1^{-1}Y^TC & -BX_1^{-1}B^T \\ kC^TC+C^TYX_1^{-1}Y^TC & A^T-C^TYX_1^{-1}B^T \end{pmatrix},$$

$X_1 := -kD^TD+D^T+D-lI$ is invertible and $Y := I-kD$.

Proof. Assume that $X_1 := -kD^TD+D^T+D-lI$ is invertible, ie: $\det(X_1)\neq 0$. Then, in the manner of [12, Lemma 1],

$$\begin{aligned}
&\det(G_1(j\omega)) \\
&= \det\left\{ \begin{pmatrix} Y^TC & -B^T \end{pmatrix} \left[j\omega I - \begin{pmatrix} A & 0 \\ -kC^TC & -A^T \end{pmatrix} \right]^{-1} \begin{pmatrix} B \\ C^TY \end{pmatrix} + X_1 \right\} \\
&= \det(X_1)\det\left\{ I + \left[j\omega I - \begin{pmatrix} A & 0 \\ -kC^TC & -A^T \end{pmatrix} \right]^{-1} \begin{pmatrix} B \\ C^TY \end{pmatrix} \begin{pmatrix} X_1^{-1}Y^TC & -X_1^{-1}B^T \end{pmatrix} \right\} \\
&= \det(X_1)\det\{(j\omega I-A)^{-1}\}\det\{(j\omega I+A^T)^{-1}\}\det(j\omega I+H_1).
\end{aligned}$$

Since A is Hurwitz (ie: has no purely imaginary eigenvalues), then $\det(j\omega I-A)\neq 0$ for all $\omega \in \mathbb{R}$. The matrix $j\omega I-A$ is invertible and so $\det\{(j\omega I-A)^{-1}\}\neq 0$. Similarly, $\det\{(j\omega I+A^T)^{-1}\}\neq 0$ noting that

$$(-1)^n\{\det(j\omega I-A)\}^* = \det(j\omega I+A^T).$$

Thus, $G_1(j\omega)$ has a zero eigenvalue if and only if $\det(j\omega I+H_1)=0$, ie: H_1 has a purely imaginary eigenvalue. Additionally, of interest are only the frequencies $\omega \in [a,b]$; correspondingly, eigenvalues of H_1 that lie on the imaginary axis between and including $-ja$ and $-jb$. $\square$

Lemma 3. *Suppose $\epsilon \in \mathbb{R}$. The matrix $-M^*(j\omega)M(j\omega) + \epsilon^2 I$ has no zero eigenvalues over $\omega \in [a,b]$ if and only if H_2 does not have any eigenvalues on the imaginary axis between and including $-ja$ and $-jb$, where*

$$H_2 := \begin{pmatrix} -A - BX_2^{-1}D^TC & -BX_2^{-1}B^T \\ C^TC + C^TDX_2^{-1}D^TC & A^T + C^TDX_2^{-1}B^T \end{pmatrix}$$

and $X_2 := -D^TD + \epsilon^2 I$ is invertible.

Proof. The proof follows in the same manner as the proof of Lemma 2, ie: assume that $X_2 := -D^TD + \epsilon^2 I$ is invertible. Then

$$\begin{aligned}
&\det(G_2(j\omega)) \\
&= \det\left\{ \begin{pmatrix} -D^TC & -B^T \end{pmatrix} \left[j\omega I - \begin{pmatrix} A & 0 \\ -C^TC & -A^T \end{pmatrix} \right]^{-1} \begin{pmatrix} B \\ -C^TD \end{pmatrix} + X_2 \right\} \\
&= \det(X_2) \det\left\{ I + \left[j\omega I - \begin{pmatrix} A & 0 \\ -C^TC & -A^T \end{pmatrix} \right]^{-1} \begin{pmatrix} B \\ -C^TD \end{pmatrix} \begin{pmatrix} -X_2^{-1}D^TC & -X_2^{-1}B^T \end{pmatrix} \right\} \\
&= \det(X_2) \det\{(j\omega I - A)^{-1}\} \det\{(j\omega I + A^T)^{-1}\} \det(j\omega I + H_2).
\end{aligned}$$

Since A is Hurwitz, then $\det\{(j\omega I - A)^{-1}\} \neq 0$ and $\det\{(j\omega I + A^T)^{-1}\} \neq 0$ for all $\omega \in \mathbb{R}$. Thus, $G_2(j\omega)$ has a zero eigenvalue if and only if $\det(j\omega I + H_2) = 0$, ie: H_2 has a purely imaginary eigenvalue. Of interest are only the frequencies $\omega \in [a,b]$; correspondingly, eigenvalues of H_2 that lie on the imaginary axis between and including $-ja$ and $-jb$. $\square$

4 A Test for "Mixedness"

Given a system state-space description as described in Section 3, we wish to determine whether or not the system is "mixed." The aim is to construct a transfer function matrix $M(s)$ from the state-space data and determine whether or not there exist $k, l > 0$ and $\epsilon < 1$ such that (i) and/or (ii) from Definition 5 hold for each frequency $\omega \in \mathbb{R}$. (Additionally, if $\lim_{\omega\to\pm\infty} \det(M^*(j\omega) + M(j\omega)) \neq 0$ and/or $\lim_{\omega\to\pm\infty} \det(-M^*(j\omega)M(j\omega) + I) \neq 0$, then we can use Remark 2 to eliminate the parameters k, l and/or ϵ from the search.) To eliminate an element of frequency-dependency from the test, however, we can choose to utilise the state-space data directly and apply Lemmas 2 and 3.

Thus, the first step of the test is to compute the matrices H_1 and H_2 (where X_1 and X_2 are constructed such that they are invertible) for $k, l \geq 0$ and $\epsilon \leq 1$ and then calculate the eigenvalues of these matrices. Existences of purely imaginary eigenvalues indicate those frequencies at which the matrices $-kM^*(j\omega)M(j\omega) + M^*(j\omega) + M(j\omega) - lI$ and $-M^*(j\omega)M(j\omega) + \epsilon^2 I$ have zero eigenvalues.

If there exist frequencies at which $-kM^*(j\omega)M(j\omega)+M^*(j\omega)+M(j\omega)-lI$ has zero eigenvalues then we divide the frequency range $(-\infty, \infty)$ up into intervals with the frequencies corresponding to the zero eigenvalues as the interval endpoints. (If there exist no frequencies at which $-kM^*(j\omega)M(j\omega) + M^*(j\omega) + M(j\omega) - lI$ has zero eigenvalues then we leave the frequency range $(-\infty, \infty)$ intact and think of it as a single "division.") Similarly, a separate set of divisions of the frequency range $(-\infty, \infty)$ can be made based on the frequencies at which $-M^*(j\omega)M(j\omega) + \epsilon^2 I$ has zero eigenvalues. (Again, the frequency range $(-\infty, \infty)$ is left intact if there exist no frequencies at which $-M^*(j\omega)M(j\omega) + \epsilon^2 I$ has zero eigenvalues.) We now have two different sets of frequency range divisions: Set of Divisions 1 and Set of Divisions 2.

Finally, we check the sign definiteness of the matrix $-kM^*(j\omega)M(j\omega)+M^*(j\omega)+M(j\omega)-lI$ over each interval belonging to Set of Divisions 1 and the sign definiteness of the matrix $-M^*(j\omega)M(j\omega) + \epsilon^2 I$ over each interval belonging to Set of Divisions 2. Testing at one frequency (eg: at the midpoint) per interval is sufficient. Those intervals over which $M^*(j\omega)+M(j\omega) > 0$ (if $\lim_{\omega\to\pm\infty} \det(M^*(j\omega) + M(j\omega)) \neq 0$; alternatively, those intervals over which $-kM^*(j\omega)M(j\omega) + M^*(j\omega) + M(j\omega) - lI \geq 0$, where $k, l > 0$) and those intervals over which $-M^*(j\omega)M(j\omega) + I > 0$ (if $\lim_{\omega\to\pm\infty} \det(-M^*(j\omega)M(j\omega) + I) \neq 0$; alternatively, those intervals over which $-M^*(j\omega)M(j\omega) + \epsilon^2 I \geq 0$, where $\epsilon < 1$) are identified and we determine whether or not there exists some combination of these intervals that spans the entire frequency range. For instance, suppose that $\lim_{\omega\to\pm\infty} \det(M^*(j\omega) + M(j\omega)) \neq 0$ and $\lim_{\omega\to\pm\infty} \det(-M^*(j\omega)M(j\omega) + I) \neq 0$. If $M^*(j\omega) + M(j\omega) > 0$ and/or $-M^*(j\omega)M(j\omega) + I > 0$ for each $\omega \in \mathbb{R}$ then the system is "mixed." If, for some $\omega \in \mathbb{R}$, both $M^*(j\omega)+M(j\omega) \not> 0$ and $-M^*(j\omega)M(j\omega) + I \not> 0$ then the system is not "mixed."

We now briefly describe the test in an algorithmic form. First, set tol_k, tol_l and tol_ϵ to some appropriate values. Tol_k and tol_l should be small (ie: close to zero) and strictly positive. Tol_ϵ should be close to and strictly less than one.

1. (a) If $\lim_{\omega\to\pm\infty} \det(M^*(j\omega) + M(j\omega)) \neq 0$:
      i. Set $k = l = 0$.
      ii. Compute H_1; calculate its eigenvalues. Denote any purely imaginary eigenvalues by $\pm j\omega_i$ and order these purely imaginary eigenvalues such that $-\omega_p \leq \cdots \leq -\omega_1 \leq \omega_1 \leq \cdots \leq \omega_p$, where $i = 1, \ldots, p$.
      iii. If H_1 has purely imaginary eigenvalues, let Set of Divisions 1 be $\{(-\infty, -\omega_p), (-\omega_p, -\omega_{p-1}), \ldots, (-\omega_1, \omega_1), \ldots, (\omega_{p-1}, \omega_p), (\omega_p, \infty)\}$. Else, let Set of Divisions 1 be $\{(-\infty, \infty)\}$. Set of Divisions 1 now contains at most $2p + 1$ frequency bands. Select a candidate testing frequency from each of these bands.
      iv. At each candidate testing frequency, compute the sign definiteness of $M^*(j\omega)+M(j\omega)$. Collect together those frequency bands containing test frequencies at which $M^*(j\omega) + M(j\omega) > 0$ into a new set of frequency bands called Revised Set of Divisions 1. Go to Step 2.

   (b) Else, if $\lim_{\omega\to\pm\infty} \det(M^*(j\omega) + M(j\omega)) = 0$:

i. Set k, l small and strictly positive (such that $k \geq \mathrm{tol}_k$ and $l \geq \mathrm{tol}_l$).

ii. Compute H_1; calculate its eigenvalues. Denote any purely imaginary eigenvalues by $\pm j\omega_i$ and order these purely imaginary eigenvalues such that $-\omega_p \leq \cdots \leq -\omega_1 \leq \omega_1 \leq \cdots \leq \omega_p$, where $i = 1, \ldots, p$.

iii. If H_1 has purely imaginary eigenvalues, let Set of Divisions 1 be $\{(-\infty, -\omega_p], [-\omega_p, -\omega_{p-1}], \ldots, [-\omega_1, \omega_1], \ldots, [\omega_{p-1}, \omega_p], [\omega_p, \infty)\}$. Else, let Set of Divisions 1 be $\{(-\infty, \infty)\}$. Set of Divisions 1 now contains exactly $2p + 1$ frequency bands. Select a candidate testing frequency from each of these bands.

iv. At each candidate testing frequency, compute the sign definiteness of $-kM^*(j\omega)M(j\omega) + M^*(j\omega) + M(j\omega) - lI$. Collect together those frequency bands containing test frequencies at which $-kM^*(j\omega)M(j\omega) + M^*(j\omega) + M(j\omega) - lI \geq 0$ into a new set of frequency bands called Revised Set of Divisions 1. Go to Step 2.

2. (a) If $\lim_{\omega \to \pm\infty} \det(-M^*(j\omega)M(j\omega) + I) \neq 0$:

i. Set $\epsilon = 1$.

ii. Compute H_2; calculate its eigenvalues. Denote any purely imaginary eigenvalues by $\pm j\omega_i$ and order these purely imaginary eigenvalues such that $-\omega_q \leq \cdots \leq -\omega_1 \leq \omega_1 \leq \cdots \leq \omega_q$, where $i = 1, \ldots, q$.

iii. If H_2 has purely imaginary eigenvalues, let Set of Divisions 2 be $\{(-\infty, -\omega_q), (-\omega_q, -\omega_{q-1}), \ldots, (-\omega_1, \omega_1), \ldots, (\omega_{q-1}, \omega_q), (\omega_q, \infty)\}$. Else, let Set of Divisions 2 be $\{(-\infty, \infty)\}$. Set of Divisions 2 now contains at most $2q + 1$ frequency bands. Select a candidate testing frequency from each of these bands.

iv. At each candidate testing frequency, compute the sign definiteness of $-M^*(j\omega)M(j\omega) + I$. Collect together those frequency bands containing test frequencies at which $-M^*(j\omega)M(j\omega) + I > 0$ into a new set of frequency bands called Revised Set of Divisions 2. Go to Step 3.

(b) Else, if $\lim_{\omega \to \pm\infty} \det(-M^*(j\omega)M(j\omega) + I) = 0$:

i. Set ϵ close to and strictly less than one (such that $\epsilon \leq \mathrm{tol}_\epsilon$).

ii. Compute H_2; calculate its eigenvalues. Denote any purely imaginary eigenvalues by $\pm j\omega_i$ and order these purely imaginary eigenvalues such that $-\omega_q \leq \cdots \leq -\omega_1 \leq \omega_1 \leq \cdots \leq \omega_q$, where $i = 1, \ldots, q$.

iii. If H_2 has purely imaginary eigenvalues, let Set of Divisions 2 be $\{(-\infty, -\omega_q], [-\omega_q, -\omega_{q-1}], \ldots, [-\omega_1, \omega_1], \ldots, [\omega_{q-1}, \omega_q], [\omega_q, \infty)\}$. Else, let Set of Divisions 2 be $\{(-\infty, \infty)\}$. Set of Divisions 2 now contains exactly $2q + 1$ frequency bands. Select a candidate testing frequency from each of these bands.

iv. At each candidate testing frequency, compute the sign definiteness of $-M^*(j\omega)M(j\omega) + \epsilon^2 I$. Collect together those frequency bands containing test frequencies at which $-M^*(j\omega)M(j\omega) + \epsilon^2 I \geq 0$ into a new set of frequency bands called Revised Set of Divisions 2. Go to Step 3.

3. Collate Revised Set of Divisions 1 and Revised Set of Divisions 2:

   (a) If Revised Set of Divisions 1 $\cup$ Revised Set of Divisions 2 $= \mathbb{R}$, then the system is "mixed." End.

   (b) Else, if $\lim_{\omega\to\pm\infty}\det(M^*(j\omega)+M(j\omega)) \neq 0$ and $\lim_{\omega\to\pm\infty}\det(-M^*(j\omega)M(j\omega)+I) \neq 0$ and Revised Set of Divisions 1 $\cup$ Revised Set of Divisions 2 $\neq \mathbb{R}$, then the system is not "mixed." End.

   (c) Else, if $\lim_{\omega\to\pm\infty}\det(M^*(j\omega)+M(j\omega)) = 0$ and Revised Set of Divisions 1 $\cup$ Revised Set of Divisions 2 $\neq \mathbb{R}$ and $k > \mathrm{tol}_k$ and/or $l > \mathrm{tol}_l$, then try a smaller, strictly positive k and/or a smaller, strictly positive l (provided that this new $k \geq \mathrm{tol}_k$ and/or this new $l \geq \mathrm{tol}_l$). Go to Step 1(b)ii. (Otherwise, go to Step 3(d).)

   (d) Else, if $\lim_{\omega\to\pm\infty}\det(-M^*(j\omega)M(j\omega)+I) = 0$ and Revised Set of Divisions 1 $\cup$ Revised Set of Divisions 2 $\neq \mathbb{R}$ and $\epsilon < \mathrm{tol}_\epsilon$, then try a larger ϵ, strictly less than one (provided that this new $\epsilon \leq \mathrm{tol}_\epsilon$). Go to Step 2(b)ii. Otherwise, make a decision on the "mixedness" of the system (eg: see Section 5, Example 3). End.

5 Examples

The following examples illustrate various aspects of the test.

Example 1. (SISO, "mixed" system) Given the state-space data $A = -5$, $B = 4$, $C = -3.25$ and $D = 3$ from which the transfer function $m_1(s)$ in Section 2 can be constructed, and setting $k = l = 0$ and $\epsilon = 1$, we obtain

$$H_1 = \begin{pmatrix} 2.8333 & -2.6667 \\ 1.7604 & -2.8333 \end{pmatrix} \text{ and } H_2 = \begin{pmatrix} 0.1250 & 2.0000 \\ -1.3203 & -0.1250 \end{pmatrix}.$$

(noting that, as $\omega \to \pm\infty$, $m_1^*(j\omega) + m_1(j\omega) \neq 0$ and $m_1^*(j\omega)m_1(j\omega) \neq 1$). The matrix H_1 does not have any purely imaginary eigenvalues which means that the sign definiteness of $m_1^*(j\omega) + m_1(j\omega)$ will remain the same over the entire frequency range $(-\infty, \infty)$. Since $m_1(j0)^* + m_1(j0) > 0$, the system is input and output strictly passive over $(-\infty, \infty)$ and is hence "mixed." We do not need to check the sign definiteness of the function $-m_1^*(j\omega)m_1(j\omega) + 1$. See Figure 3 for an illustration of the system's frequency response.

Example 2. (SISO system, not "mixed") Given the state-space data $A = -4$, $B = 4$, $C = -2.75$ and $D = 2$ from which the transfer function $m_4(s)$ in Section 2 may be constructed, and setting $k = l = 0$ and $\epsilon = 1$, we obtain

$$H_1 = \begin{pmatrix} 1.25 & -4 \\ 1.890625 & -1.25 \end{pmatrix} \text{ and } H_2 = \begin{pmatrix} -3.\dot{3} & 5.\dot{3} \\ -2.52083\dot{3} & 3.\dot{3} \end{pmatrix}$$

(noting that, as $\omega \to \pm\infty$, $m_4^*(j\omega) + m_4(j\omega) \neq 0$ and $m_4^*(j\omega)m_4(j\omega) \neq 1$). The matrix H_1 has two purely imaginary eigenvalues, $\pm j2.4495$. Breaking the frequency range $(-\infty, \infty)$ up

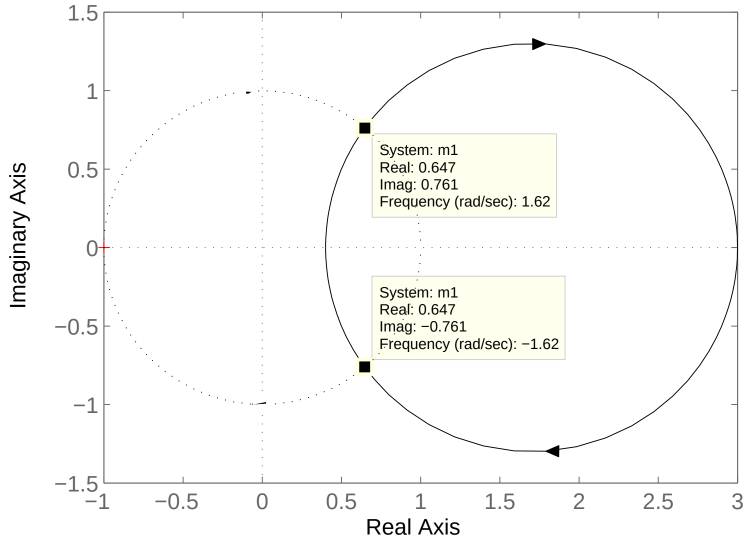

Figure 3: Nyquist diagram of $m_1(s)$.

into the intervals $(-\infty, -2.4495)$, $(-2.4495, 2.4495)$ and $(2.4495, \infty)$ and examining the sign definiteness of $m_4^*(j\omega) + m_4(j\omega)$ at some single frequency point from within each of these intervals (eg: at $\omega = -4, 0, 4$) yields $m_4(-j4)^* + m_4(-j4) > 0$, $m_4(j0)^* + m_4(j0) \not> 0$ and $m_4(j4)^* + m_4(j4) > 0$. Thus, the system is passive over $(-\infty, -2.4495]$ and $[2.4495, \infty)$ and a system gain of less than one over $[-2.4495, 2.4495]$ is required in order for it to be "mixed." This requirement is not met, as follows.

The matrix H_2 has two purely imaginary eigenvalues, $\pm j1.5275$. Observing the sign definiteness of $-m_4^*(j\omega)m_4(j\omega) + 1$ at a single frequency point from within each of the intervals $(-\infty, -1.5275)$, $(-1.5275, 1.5275)$ and $(1.5275, \infty)$ (eg: at $\omega = -2, 0, 2$) yields $-m_4(-j2)^* m_4(-j2) + 1 \not> 0$, $-m_4(j0)^* m_4(j0) + 1 > 0$ and $-m_4(j2)^* m_4(j2) + 1 \not> 0$. Thus, the system has a gain of less than or equal to one over the frequency interval $[-1.5275, 1.5275]$ as opposed to a gain of less than one over the (larger) frequency interval $[-2.4495, 2.4495]$ (that is, there exist frequencies at which both $m_4^*(j\omega) + m_4(j\omega) \not> 0$ and $-m_4^*(j\omega)m_4(j\omega) + 1 \not> 0$). Hence, the system is not "mixed." See Figure 4 for an illustration of the system's frequency response.

Example 3. (SISO, "mixed" strictly proper system) Consider the state-space description

$$A = \begin{pmatrix} -3 & -2 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 1.5 \end{pmatrix}, D = 0$$

from which the strictly proper transfer function $m_3(s)$ in Section 2 can be constructed.

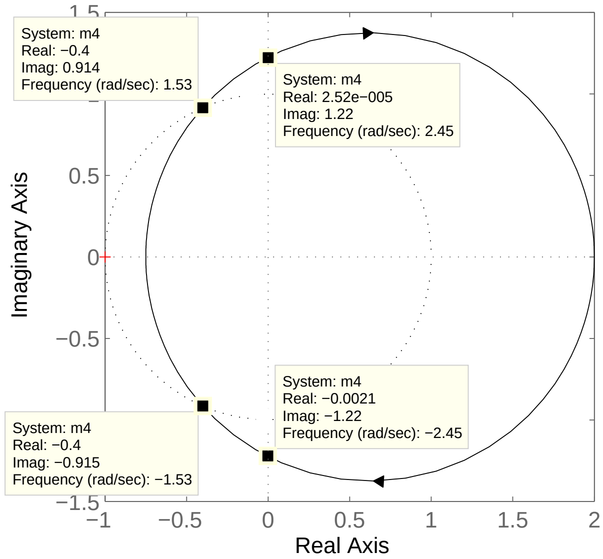

Figure 4: Nyquist diagram of $m_4(s)$.

Setting $\epsilon = 1$, we obtain

$$H_2 = \begin{pmatrix} 3 & 2 & -4 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 2.25 & -2 & 0 \end{pmatrix}$$

which has two purely imaginary eigenvalues, $\pm j0.924$. Breaking the frequency range $(-\infty, \infty)$ up into the intervals $(-\infty, -0.924)$, $(-0.924, 0.924)$ and $(0.924, \infty)$ and examining the sign definiteness of $-m_3^*(j\omega)m_3(j\omega) + 1$ at a single frequency point from within each of these intervals (eg: at $\omega = -1, 0, 1$) yields $-m_3(-j1)^* m_3(-j1) + 1 > 0$, $-m_3(j0)^* m_3(j0) + 1 \ngtr 0$ and $-m_3(j1)^* m_3(j1) + 1 > 0$. Thus, the system does not have a gain of less than one over $[-0.924, 0.924]$ and must be input and output strictly passive over this interval if it is to be "mixed."

The difficulty with setting $k = l = 0$ and applying Lemma 2 to determine the zeros of the function $m_3^*(j\omega) + m_3(j\omega)$ is that the system is strictly proper and hence $D^T + D = 0$ is not invertible. In this example, the difficulty is overcome by setting k and l to be decreasingly smaller values but not equal to zero which, by continuity, will provide us with an indication of those frequencies at which the system isn't input and output strictly passive anymore but is passive. First, note that $m_3(j0)^* + m_3(j0) > 0$. Table 1 lists the frequencies at which $-km_3^*(j\omega)m_3(j\omega) + m_3^*(j\omega) + m_3(j\omega) - lI$ has zero eigenvalues (ie: the frequencies which correspond to the purely imaginary eigenvalues of H_1) for decreasingly smaller values of k and l. As $k, l \to 0$, these frequencies approach ± 1.414 which indicates that the system is input and output strictly passive over $(-1.414, 1.414)$. Since $[-0.924, 0.924]$ is a subset of

k	l	ω_{H_1}
0.01	0.01	± 1.398
0.001	0.001	± 1.413
0.0001	0.0001	± 1.414

Table 1: Zeros of $G_1(j\omega)$.

$(-1.414, 1.414)$ then the system is "mixed." The frequency response of the system is depicted in Figure 5.

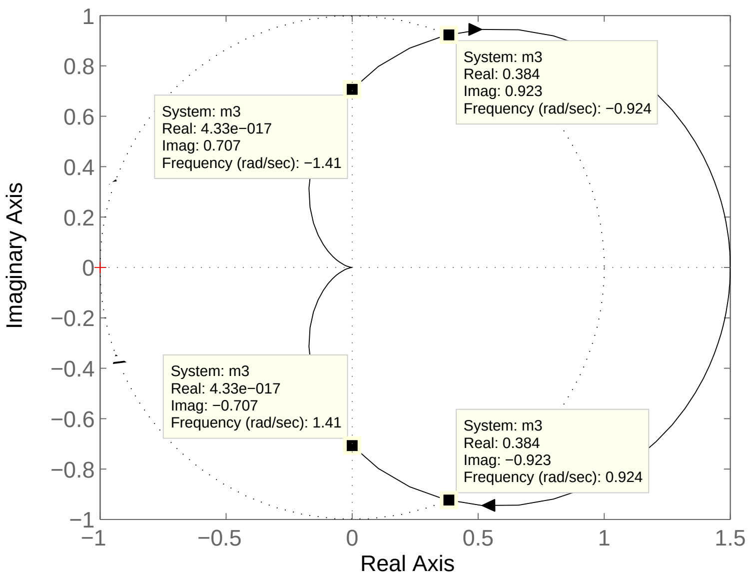

Figure 5: Nyquist diagram of $m_3(s)$.

Example 4. (MIMO system, not "mixed") Given the state-space data

$$A = \begin{pmatrix} 0 & -2 & 0 & 0 \\ 1 & -3 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, B = \begin{pmatrix} 0.75 & 2.5 \\ 0 & 0 \\ 4 & 0 \\ 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 0 & 4 & 0 & 0 \\ 0 & 0 & -3.25 & 1 \end{pmatrix}$$

and

$$D = \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix}$$

from which the transfer function matrix

$$M(s) = \begin{pmatrix} \frac{3}{(s+1)(s+2)} & \frac{10}{(s+1)(s+2)} \\ \frac{3s+2}{s+5} & \frac{1}{s+1} \end{pmatrix}$$

can be constructed, and setting $k = l = 0$ and $\epsilon = 1$, we obtain

$$H_1 = \begin{pmatrix} 0 & 5.3333 & -0.8125 & 0.25 & -1.25 & 0 & -3.3333 & -0.25 \\ -1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.6667 & 1.3333 & -3.3333 & 0 & 0 & -1.3333 \\ 0 & 1.3333 & 0 & 1 & -0.25 & 0 & -1.3333 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -4.3333 & 1.3333 & -5.3333 & -3 & 0 & -1.3333 \\ 0 & -4.3333 & 0 & 0 & 0.8125 & 0 & -0.6667 & 0 \\ 0 & 1.3333 & 0 & 0 & -0.25 & 0 & -1.3333 & -1 \end{pmatrix}$$

and

$$H_2 = \begin{pmatrix} 0 & 2 & -0.9141 & 0.2813 & -6.1797 & 0 & 0.375 & -2.5 \\ -1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.125 & 1.5 & 0.375 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & -2.5 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 16 & 0 & 0 & -2 & -3 & 0 & 0 \\ 0 & 0 & -1.3203 & 0.4063 & 0.9141 & 0 & -0.125 & 0 \\ 0 & 0 & 0.4063 & -0.125 & -0.2813 & 0 & -1.5 & -1 \end{pmatrix}$$

(noting that $\lim_{\omega\to\pm\infty} \det(M^*(j\omega)+M(j\omega)) \neq 0$ and $\lim_{\omega\to\pm\infty} \det(-M^*(j\omega)M(j\omega)+I) \neq 0$). The matrix H_1 does not have any purely imaginary eigenvalues which means that the sign definiteness of $M^*(j\omega) + M(j\omega)$ is the same over the entire frequency range $(-\infty, \infty)$. Since $M(j0)^* + M(j0) \not> 0$, we require that the system has a gain of less than one over $(-\infty, \infty)$ in order for it to be "mixed." Unfortunately, this is not the case. The matrix H_2 has four purely imaginary eigenvalues, $\pm j1.67$ and $\pm j2.72$. Breaking the frequency range $(-\infty, \infty)$ up into the intervals $(-\infty, -2.72)$, $(-2.72, -1.67)$, $(-1.67, 1.67)$, $(1.67, 2.72)$ and $(2.72, \infty)$ and examining the sign definiteness of $-M^*(j\omega)M(j\omega) + I$ at a single frequency point from within each of these intervals (eg: at $\omega = -3, -2, 0, 2, 3$) yields that $-M(j0)^*M(j0) + I$ is an indefinite matrix, for example, and thus that the system is not "mixed."

6 Conclusions

A necessary and sufficient test for determining whether or not a causal, stable, MIMO, LTI system is "mixed" was developed. Implementation of the test is based on determining the purely imaginary eigenvalues of Hamiltonian matrices.

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