A Result on the Existence of Quadratic Lyapunov Functions for State-Dependent Switched Systems with Uncertainty

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***Abstract*— Recent results on quadratic stability of state-dependent switched linear systems are reviewed and applied to the problem of state-dependent switching with parameter uncertainty. Several examples are provided to illustrate the method, both as a means of determining stability of such systems, and also as a tool for designing stable switched systems in the presence of uncertainty.**

I. Introduction

We consider the problem of designing state-dependent switching systems subject to parameter uncertainty. Unlike many of the problems considered in the switched systems literature, the switching rule considered in this paper is governed by a *fixed partition* of the state-space which cannot be manipulated in order to achieve a stabilising switching law [1]. To be specific, we consider models where the state-space is partitioned into two regions, namely a closed double convex pointed cone in $\mathbb{R}^n$ and its complement, with different dynamics operating in each region, and where the goal is to find criteria that will ensure the stability of the resulting nonlinear system. Such problems may arise, for example, in rollover prevention systems in automotive control where an emergency action is initiated if the roll angle exceeds some critical value and the load transfer ratio is greater than unity [2]. Furthermore the dynamics operating in each region are subject to parameter uncertainty. Stability of this system class will be analysed by invoking a result that determines the existence of a quadratic Lyapunov function (QLF) which is decreasing along every trajectory of the system [3]. The result is illustrated by means of several examples.

II. Main Result

Our basic objective is to deduce exponential stability of the switched system

$$\dot{x} = A(x)x, \;\; A(x) \in \{A, B\}, \tag{1}$$

where A and B are Hurwitz matrices, where switching is orchestrated by a partition of the state-space that is fixed a-priori, and where the matrix B is subject to parameter uncertainty. Our specific objective here is to consider the case when B is a rank-1 perturbation of A:

$$B = A - bc^T$$

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for some $b, c \in \mathbb{R}^n$, where (A, b) is controllable and (A, c) is observable. Parameter uncertainty arises through uncertainty in the location of the vector c. Our method is to use a quadratic Lyapunov function to deduce the stability of the nonlinear system (1). The partition of $\mathbb{R}^n$ which governs switching has the following specific form. Let $\mathcal{C}$ be a convex polyhedral cone in $\mathbb{R}^n$, and let $\mathcal{C}^+$ be its positive polar (also known as the dual cone) defined by $\mathcal{C}^+ = \{y : y^T x \geq 0 \,\text{for all}\, x \in \mathcal{C}\}$. Then $\mathcal{C}^+$ is also a convex polyhedral cone [4]. We will assume that c is in $\mathcal{C}^+$, that is

$$c^T x \geq 0 \quad \text{for all } x \in \mathcal{C}. \tag{2}$$

The region Ω is defined as $\Omega = \mathcal{C} \cup (-\mathcal{C})$. Then we wish to determine the existence of a matrix $P = P^T > 0$ such that $A^T P + PA < 0$ and $x^T \left(B^T P + PB\right) x < 0$ for all $x \neq 0 \in \Omega$.

Before proceeding we note that such problems are well motivated and arise frequently in practice, and often more general partitions of $\mathbb{R}^n$ can be reformulated in this manner. To make this point clearer, we depict in the plane the type of switching problem that is of interest (see Figs. 3 and 8). Recall, we wish to design a dynamic system of the form

$$\dot{x} = \begin{cases} Ax & x \in \mathbb{R}^n \\ (A - bc^T)x & x \in \Omega \end{cases}$$

such that the closed loop is exponentially stable, where the region Ω is symmetric with respect to the origin. Such systems arise where the control system has a normal mode of operation; and an emergency mode that may be activated based on some other external signal. Another less obvious, but perhaps more compelling motivation, for addressing this type of model arises in situations where the switching hyperplanes do not pass through the origin. Such a situation is depicted graphically in Fig. 1. In Fig. 1, the maximum and minimum of x_2 might for example represent actuator constraints, and the threshold on x_1 the actual switching logic. In this situation, we might look for a QLF $V(x) = x^T P x$ satisfying

$$x^T(PA + A^T P)x < 0 \quad x \in K$$
$$x^T(P(A - bc^T) + (A - bc^T)^T P)x < 0 \;\; x \in \mathbb{R}^n \setminus K.$$

However since K contains the origin, scaling invariance requires that $x^T(PA + A^T P)x < 0$ for all $x \in \mathbb{R}^n$. Similar considerations imply that $x^T(P(A-bc^T)+(A-bc^T)^T P)x < 0$ for all $x \in \Omega$, where Ω is the smallest subset of $\mathbb{R}^n$ which contains $\mathbb{R}^n \setminus K$ and is invariant under re-scaling $x \mapsto \lambda x$ for all $\lambda \neq 0$. Thus the question of existence of a quadratic

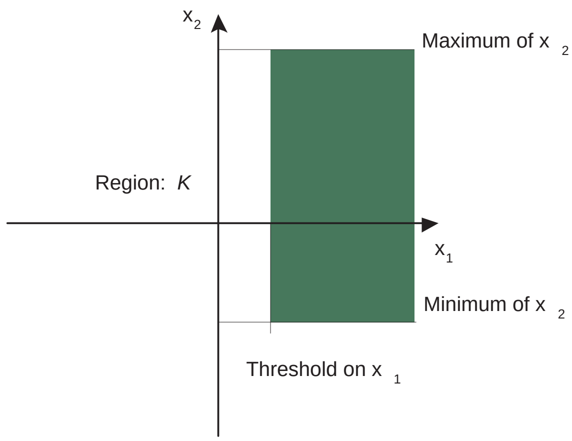

Fig. 1. Partitioning of the state space.

Lyapunov function for such a switched system requires the solution of the problem addressed in this paper. This is depicted in Fig. 2. Note that this argument applies to any state-dependent switching problem where we are studying linear systems using a Lyapunov function $V(x)$ and where $\nabla V(rx)$ is parallel to $\nabla V(x)$ for all $r > 0$ and all $x \in \mathbb{R}^n$. We then have the following result [3].

Theorem 1: Let $\Omega = \mathcal{C} \cup (-\mathcal{C})$ where $\mathcal{C}$ is a convex polyhedral cone satisfying (2). The following conditions are equivalent:

1) there exists a positive definite matrix P satisfying $PA + A^T P < 0$ and the constraint condition

$$\langle Px, Ax \rangle < \langle Px, b \rangle \langle c, x \rangle \text{ for all nonzero } x \in \Omega; \quad (3)$$

2) there is a vector $v \in \mathcal{C}^+$ such that

$$1 + \mathrm{Re}\,(c + v)^T (j\omega I - A)^{-1} b > 0 \quad (4)$$

for all $\omega \in \mathbb{R}$.

A full proof of this result in given in [3]. Here we briefly note that the derivation borrows from, and extends, our previous results on this topic. In particular many similar ideas can be found in [5], [6].

Remark 1: The condition (4) is a generalisation of the frequency condition of the Kalman-Yakubovich-Popov (KYP) lemma. Indeed, suppose that $\Omega = \mathbb{R}^n$, then $\Omega = \mathcal{C} \cup -\mathcal{C}$ where $\mathcal{C} = \{x : x^T c \geq 0\}$. In this case $\mathcal{C}^+ = \{\lambda c : \lambda \geq 0\}$, and so the condition $v \in \mathcal{C}^+$ in Condition 2 of the theorem is equivalent to $v = \lambda c$ for some $\lambda \geq 0$. Thus the frequency condition (4) becomes

$$1 + (1 + \lambda)\,\mathrm{Re}\; c^T (j\omega I - A)^{-1} b > 0$$

for all $\omega \in \mathbb{R}$. This implies the usual KYP condition, and hence in this case the result of Theorem 1 reduces to the classical KYP theorem.

Remark 2: The classical version of the KYP lemma has been extended many times. For instance, starting with the paper of [7], several authors have proposed new constrained formulations. Of particular note in this direction is a recent paper by Iwasaki and Hara [8] in which the authors generalise the lemma by investigating the manner in which constraints on the frequency domain inequality (FDI) affect the corresponding linear matrix inequality (LMI). Our main result is of a similar spirit to that of Iwasaki and Hara; however, motivated by a problem in the design of switched systems, we impose constraints on the LMI and investigate the consequences of these on the FDIs.

To summarise, the main result says that there is a joint quadratic Lyapunov function (JQLF) for $(A, A - bc^T)$ in the region Ω if and only if there is a common quadratic Lyapunov function (CQLF) for the pair $(A, A - b(c + v)^T)$ where v is some vector in $\mathcal{C}^+$. Namely,

(a) (**JQLF**) there exists a positive definite $P = P^T > 0$ such that $A^T P + PA < 0$ and $x^T \left((A - bc^T)^T P + P(A - bc^T)\right) x < 0$ for all nonzero $x \in \Omega$;

(b) (**CQLF**) if and only if there exists a positive definite $P_1 = P_1^T > 0$ such that $A^T P_1 + P_1 A < 0$ and $(A - bw^T)^T P_1 + P_1(A - bw^T) < 0$ where $w = c + v$ and c, v are in $\mathcal{C}^+$.

If the positivity condition (2) is not satisfied in the cone $\mathcal{C}$ then it is still possible to find a constrained multi-dimensional frequency domain condition giving necessary and sufficient conditions for the existence of P, but the constraints become more complicated and onerous to check.

III. Examples: Systems Without Parameter Uncertainty

We now illustrate the use of our main result by means of several examples where the parameters are fixed and known. First we illustrate a case where no JQLF exists.

Example 1 (Nonexistence of P for $n = 2$): Suppose that

$$A = \begin{pmatrix} -0.3 & 1.8 \\ -1 & 0.2 \end{pmatrix}, B = \begin{pmatrix} -1.3 & -0.2 \\ -1.3 & -0.4 \end{pmatrix}$$

and $B = A - bc^T$, where $b^T = \begin{pmatrix} 1 & 0.3 \end{pmatrix}$ and $c^T = \begin{pmatrix} 1 & 2 \end{pmatrix}$. Let $x_1^T = \begin{pmatrix} -1 & 0.9 \end{pmatrix}$ and $x_2^T = \begin{pmatrix} -0.5 & 1 \end{pmatrix}$ and let the region $\mathcal{C}$ be characterised by the set of vectors

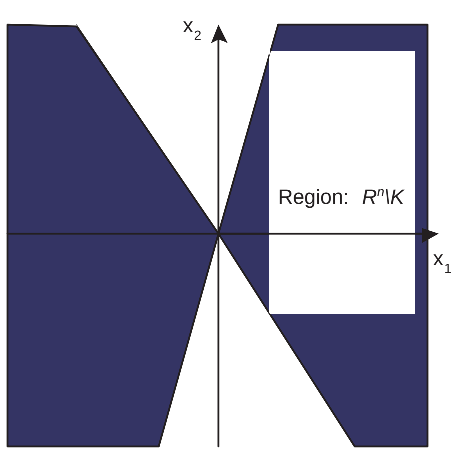

Fig. 2. Equivalent partitions as in Fig. 1 taking into account scale invariance of the linear system and symmetry of the Lyapunov function. The region Ω is the entire darkened region.

$\{\alpha x_1 + \beta x_2 \,|\, \alpha, \beta \in \mathbb{R} \text{ and } \alpha, \beta \geq 0\}$. Let $\Omega = \mathcal{C} \cup -\mathcal{C}$, noting that the boundary of Ω is the pair of lines parallel to x_1 and x_2 passing through the origin, as depicted in Fig. 3. The positivity condition (2) is satisfied by this construction. From the main result, a necessary and sufficient condition for the existence of a positive definite matrix P which satisfies $A^T P + PA < 0$ and the constraint condition (3) is that there exists a vector $v \in \mathcal{C}^+$ such that

$$1 + \operatorname{Re} c^T (j\omega I - A)^{-1} b + \operatorname{Re} v^T (j\omega I - A)^{-1} b > 0$$

for all $\omega \in \mathbb{R}$. Since the positive linear span of the two vectors $h_1^T = \begin{pmatrix} 0.9 & 1 \end{pmatrix}$ and $h_2^T = \begin{pmatrix} -1 & -0.5 \end{pmatrix}$ is the dual cone, ie: $\mathcal{C}^+ = \{\delta_1 h_1 + \delta_2 h_2 \,|\, \delta_1, \delta_2 \in \mathbb{R} \text{ and } \delta_1, \delta_2 \geq 0\}$, then we can rewrite the necessary and sufficient condition as there exist some real constants $\delta_1, \delta_2 \geq 0$ such that

$$1 + \operatorname{Re} c^T (j\omega I - A)^{-1} b + \sum_{a=1}^{2} \delta_a \operatorname{Re} h_a^T (j\omega I - A)^{-1} b > 0 \quad (5)$$

for all $\omega \in \mathbb{R}$. Substituting A, b, c and h_a (where $a = 1, 2$) into (5) and using the fact that $\operatorname{Re} c^T (j\omega I - A)^{-1} b = -c^T (A^2 + \omega^2 I)^{-1} A b$ gives

$$y_0 + \delta_1 y_1 + \delta_2 y_2 > 0, \quad (6)$$

where

$$\begin{aligned} y_0 :=& 1 + \operatorname{Re} c^T (j\omega I - A)^{-1} b \\ =& 1 - \left(\frac{-1.64\omega^2 + 2.5752}{\omega^4 - 3.47\omega^2 + 3.0276} \right) \\ y_1 :=& \operatorname{Re} h_1^T (j\omega I - A)^{-1} b = -\left(\frac{-0.724\omega^2 + 1.05096}{\omega^4 - 3.47\omega^2 + 3.0276} \right) \\ y_2 :=& \operatorname{Re} h_2^T (j\omega I - A)^{-1} b = -\left(\frac{0.23\omega^2 - 0.2001}{\omega^4 - 3.47\omega^2 + 3.0276} \right). \end{aligned}$$

The functions of frequency y_0, y_1 and y_2 are shown in Figs. 4, 5 and 6, respectively. (A magnified section of y_2 is depicted in Fig. 7.) From the figures, it is clear that at $\omega = 1$, for instance, no combination of $\delta_1, \delta_2 > 0$ can exist such that (5) holds (since, at this frequency, $y_0, y_1, y_2 < 0$). Thus, a positive definite matrix P satisfying the required constraints does not exist.

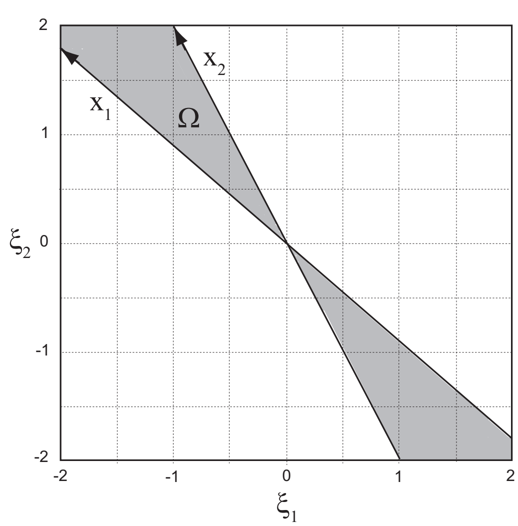

Fig. 3. Example 1: partitioning of the state space.

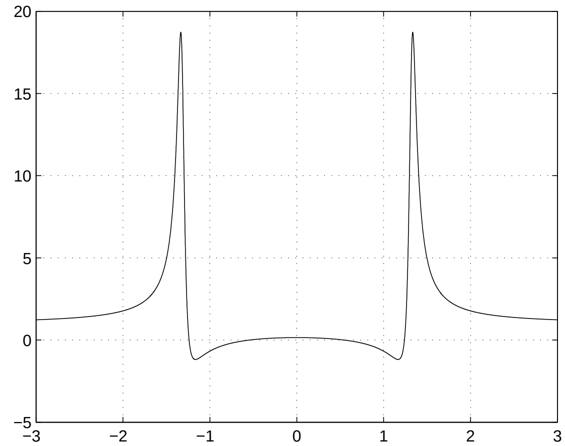

Fig. 4. The function y_0.

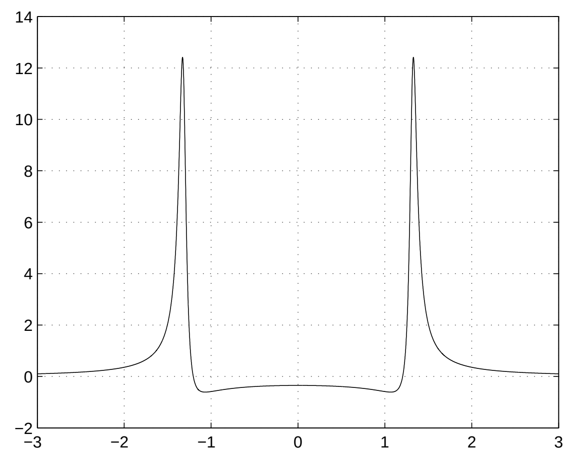

Fig. 5. The function y_1.

Now we give an example where a JQLF exists.

Example 2 (Existence of P for $n = 2$): Consider A, B, b and c as given in Example 1. Suppose, however, that the region Ω is defined by the vectors $x_1^T = \begin{pmatrix} 0.1 & 1 \end{pmatrix}$ and $x_2^T = \begin{pmatrix} 1 & 0.1 \end{pmatrix}$, as shown in Fig. 8, using a construction which is otherwise the same as that in Example 1. The positive linear span of the vectors $h_1^T = \begin{pmatrix} -0.1 & 1 \end{pmatrix}$ and $h_2^T = \begin{pmatrix} 1 & -0.1 \end{pmatrix}$ is now the dual cone and so a necessary and sufficient condition for the existence of a positive definite matrix P which satisfies $A^T P + PA < 0$ and the constraint condition (3) is that there exist some real constants $\delta_1, \delta_2 \geq 0$

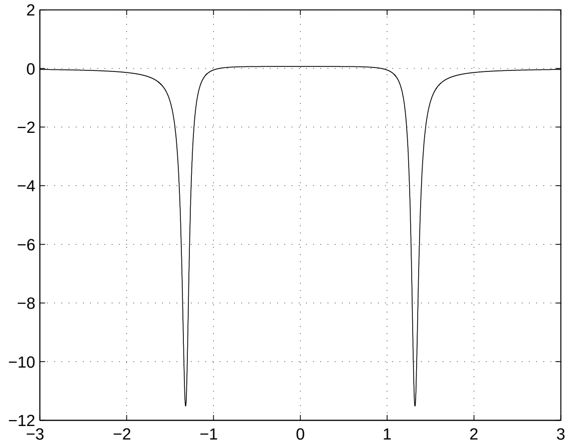

Fig. 6. The function y_2.

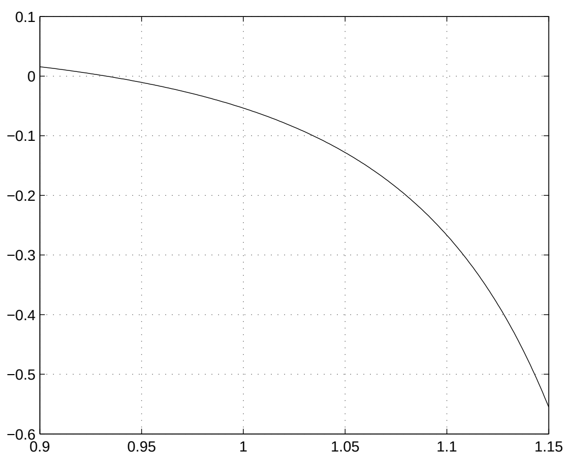

Fig. 7. A magnified section of the function y_2.

such that (6) holds for all $\omega \in \mathbb{R}$, where

$$y_1 := \mathrm{Re}\, h_1^T (j\omega I - A)^{-1} b = -\left(\frac{0.334\omega^2 - 0.74994}{\omega^4 - 3.47\omega^2 + 3.0276} \right)$$

$$y_2 := \mathrm{Re}\, h_2^T (j\omega I - A)^{-1} b = -\left(\frac{-0.964\omega^2 + 1.64256}{\omega^4 - 3.47\omega^2 + 3.0276} \right)$$

and y_0 is the same as in Example 1. The new functions of frequency y_1 and y_2 are shown in Figs. 9 and 10, respectively.

In this case, since $y_1 > 0$ over those frequency intervals where $y_0 < 0$, a δ_1 may be chosen such that (6) holds (where δ_2 is chosen appropriately). Thus, a positive definite matrix P satisfying the required constraints must exist. Indeed, such a P is

$$\begin{pmatrix} 1 & -0.25 \\ -0.25 & 1.8 \end{pmatrix}.$$

IV. Uncertain Switched Systems

Theorem 1 can be interpreted as saying that a JQLF exists for the state-dependent switched system if a related non-

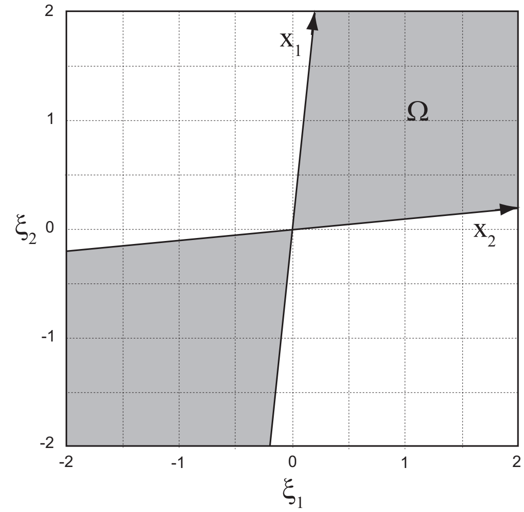

Fig. 8. Example 2: partition of the state space.

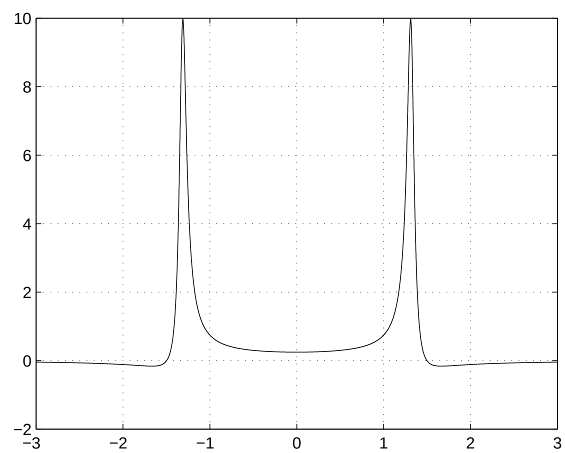

Fig. 9. The new function y_1.

state-dependent switched system has a CQLF. Since one must search over the entire cone $\mathcal{C}^+$ to find this related system, at first glance this result may appear to be difficult to use as a stability verification tool. Indeed, one might ask if it were not in fact preferable to solve the problem of interest using the S-procedure in conjunction with an LMI solver [3], [9]. However, apart from the theoretical insights to be gleaned from our results, Theorem 1 can provide the basis for an effective design procedure for uncertain switched systems, as we now describe. To see this we now interpret Theorem 1 in a slightly different manner. Recall that Theorem 1 states:

"A JQLF exists if one can find a vector v in the positive polar $\mathcal{C}^+$, such that the pair $A, A - b(c+v)^T$ has a CQLF."

Suppose that this condition is satisfied, with $c + v = w$. Now suppose that the entries of the vector c are uncertain and that the true vector is in fact $\tilde{c}$. By Theorem 1 we know that a JQLF will exist provided we can find $\tilde{v}$ in $\mathcal{C}^+$ such that $\tilde{c} + \tilde{v} = w$. In this way we can use the dual cone in a constructive manner to build an uncertainty region around

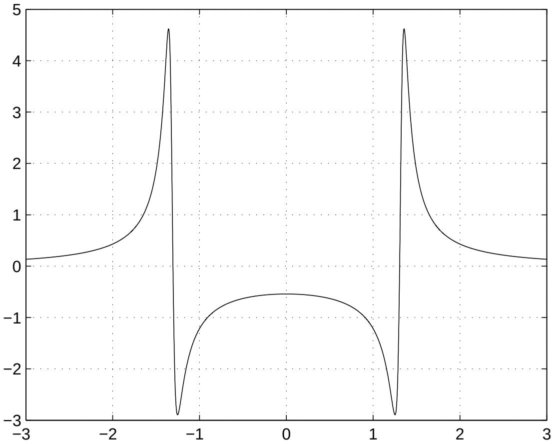

Fig. 10. The new function y_2.

c such that a JQLF exists for the true, but unknown, fixed parameter $\tilde{c}$. To see this consider the following control design strategy.

(A) Given c, pick a $v \in \mathcal{C}^+$. Set $w = c + v$.
(B) Now pick A (assuming it exists) such that both A and $A - bw^T$ are Hurwitz and have a CQLF, and such that $A - bc^T$ is also Hurwitz.
(C) Now consider a perturbation of the vector $c : \tilde{c} = c + \delta c$. The set of vectors $\tilde{v} \in \mathcal{C}^+$ such that $\tilde{c} + \tilde{v} = w$ defines an uncertainty region for the vector c such that a JQLF exists for the original (uncertain) switched system provided $A - b\tilde{c}^T$ is Hurwitz in the uncertainty region. Tools to ensure this latter requirement can be borrowed from robust stability theory.

The following examples illustrate the above observation.

Example 3: Suppose that $x_1^T = \begin{pmatrix} 1 & 0 \end{pmatrix}$ and $x_2^T = \begin{pmatrix} 1 & 1 \end{pmatrix}$ so that the region $\mathcal{C}$ is characterised by the set of vectors $\{\alpha x_1 + \beta x_2 \,|\, \alpha, \beta \in \mathbb{R}$ and $\alpha, \beta \geq 0\}$ and $\Omega = \mathcal{C} \cup -\mathcal{C}$. The positive linear span of the vectors $h_1^T = \begin{pmatrix} 0 & 1 \end{pmatrix}$ and $h_2^T = \begin{pmatrix} 1 & -1 \end{pmatrix}$ is the dual cone, ie: $\mathcal{C}^+ = \{\gamma h_1 + \eta h_2 \,|\, \gamma, \eta \in \mathbb{R}$ and $\gamma, \eta \geq 0\}$. Let

$$A = \begin{pmatrix} 12 & -20 \\ 12 & -17 \end{pmatrix}, B = \begin{pmatrix} -16 & -34 \\ 2 & -22 \end{pmatrix},$$

$b^T = \begin{pmatrix} 28 & 10 \end{pmatrix}$ and $c^T = \begin{pmatrix} 1 & 0.5 \end{pmatrix}$, where the matrices A and B are Hurwitz, $B = A - bc^T$ and the positivity condition (2) is satisfied. Note that there does not exist a positive definite matrix P which satisfies $A^T P + PA < 0$ and $B^T P + PB < 0$ since the matrix AB has real negative eigenvalues; however, there exists a positive definite matrix P which satisfies $A^T P + PA < 0$ and the constraint condition (3) since, for instance, if we choose $v^T = \begin{pmatrix} 1 & -0.8 \end{pmatrix} \in \mathcal{C}^+$, then $A(A - (c+v)^T)$ has no real negative eigenvalues. Suppose that there is some uncertainty associated with the bottom entry of the vector c, denoted $\tilde{c}^T = [\tilde{c}_1, \tilde{c}_2]$. Then, we also may deduce the existence of a JQLF for the original system provided we can find another $\tilde{v} \in \mathcal{C}^+$ such that $\tilde{v}_2 + \tilde{c}_2 = -0.3$, and provided that the matrix $A - b\tilde{c}^T$ is Hurwitz stable over the entire uncertainty range.

Example 4: Suppose that we wish to design an exponentially stable switched system where $\mathcal{C}$ is the closed nonnegative orthant in $\mathbb{R}^5$, and where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & -4 & -9 & -9 & -4 \end{bmatrix}$$

with the vector b fixed and $b^T = [0.1, 0.1, 0.1, 0.1, 0.1]$. Note that $\mathcal{C}^+ = \mathcal{C}$ in this case. Suppose now that the vector c is uncertain; that is each entry in the vector c satisfies $\underline{c}_i < c_i < \bar{c}_i$. For example we may take $0 < c_i < 1$ for $i = 1, 2$ and $0 < c_i < 2$ otherwise. We wish to ensure that a joint Lyapunov function exists for the system irrespective of the unknown (but fixed) value of c. We achieve this by invoking our main result. From our main theorem, we know that the system

$$\dot{x} = A(x)x, A(x) \in \{A, A - bc^T\}, \tag{7}$$

will have a JQLF provided that for some $v \in \mathcal{C}$ the matrix product $A\big(A - b(c+v)^T\big)$ has no negative eigenvalue. Since c is bounded above by $w^T = [1, 1, 2, 2, 2]$ we know that the vector $v = w - c$ lies in $\mathcal{C}$ for any uncertain c satisfying the above inequalities. Furthermore, it is easily verified that both A and $A - bw^T$ are Hurwitz stable, and that the product of these matrices $A(A - bw^T)$ has no negative eigenvalues. Thus the conditions of our theorem are satisfied, and we can apply the results to deduce existence of a joint Lyapunov function. Thus, the following statements are true.

(i) The dynamic systems $\dot{x} = Ax$ and $\dot{x} = (A - bw^T)x$ have a CQLF.
(ii) The system (7) is exponentially stable for all fixed vectors c which lie in the uncertainty range and for which $A - bc^T$ is Hurwitz.

For example, note that for $c^T = [0.99, 0.99, 0, 0.99, 2]$, the matrix $A - bc^T$ is Hurwitz and the matrix $A(A - bc^T)$ has negative eigenvalues. Thus it follows that no CQLF exists for $\dot{x} = Ax$ and $\dot{x} = (A - bc^T)x$ whereas a JQLF does exist for the original system.

V. Related conditions in the literature

An interesting observation regarding Theorem 1 is that it gives a solution to a state dependent switching problem (for which there is no common quadratic Lyapunov function), in terms of the existence of a common quadratic Lyapunov function for a related problem. This seems a very strange result. Our main purpose here in this section is to note that such reductions, although not well known, are a feature of classical stability theory. An example of another result taking this form is the classical SISO Popov criterion [10]. The Popov criterion arises in the study of Lure' systems:

$$\begin{aligned} \dot{x} &= Ax + bu, \\ y &= c^T x, \\ u &= -k(y) \end{aligned}$$

where A is an $n \times n$ matrix, b, c are vectors, u is a scalar and where the function $k(y)$ is nonlinear but time-invariant. A sufficient condition for the absolute stability of this system (via a Lure'-Postnikov Lyapunov function and assuming $k(y)$ is sector bounded in $(0, 1)$) is that A and $A - bc^T$ are Hurwitz and there exists a strictly positive $\alpha \in \mathbb{R}$ such that

$$1 + Re\left\{(1 + j\alpha\omega)c^T(j\omega I_n - A)^{-1}b\right\} \quad > \quad 0 \;\forall\; \omega \in \mathbb{R}.$$

Suppose now that the transfer function is strictly proper. Then, it follows from the results in [11] that a time-domain version of the Popov criterion can be obtained from the positive real condition. This condition is equivalent to requiring that the systems:

$$\dot{x} = Ax; \;\; \dot{x} = B(\alpha)x, \tag{8}$$

have a CQLF, where $B(\alpha)$ is a Hurwitz matrix constructed using c, b, A and α. As with our main result, the original stability question has been reduced to a CQLF existence problem for a related switched linear system. This appears to be an unexplored (and perhaps important) observation as it suggests that sometimes, complicated stability problems can be solved by replacing the original nonlinear system with a related switched system, the quadratic stability of which implies the non-quadratic stability of the original system.

VI. Conclusions

In this paper, we presented a solution to the stability problem for a switched system with parameter uncertainty, by applying a result which is an extension of the classical KYP lemma. While previous work in this direction has focused mainly on the implications for the LMI when one constrains the FDI in the KYP lemma, in our model we constrain the LMI and investigate the implications of this constraint for the FDI, taking into account the additional effect of parameter uncertainty. In our example the nonnegative orthant was used to construct the switching region, however our main result is applicable to problems where switching is governed by any convex polyhedral cone.

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