Determining "Mixedness" and an Application of Finite-Gain Stability Results to "Mixed" System Interconnections

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***Abstract*— A loss of passivity in the face of certain frequency dynamics (eg: high frequency dynamics) given an otherwise passive system leads to the notion of a "mixed" system. A "mixed" system is one that has a concept of small gain associated with it over those frequency intervals where passivity is lost. In this paper, a test for determining "mixedness" for linear, time-invariant systems is provided and several finite-gain stability results for interconnections of such systems are presented. The "mixedness" test involves examining the spectral characteristics of two Hamiltonian matrices, one associated with the passive aspects of the system and one associated with the notion of small gain. The finite-gain stability results are derived using a dissipative systems framework.**

I. Introduction

The concept of a "mixed" system arises from a need to deal with situations where the passivity properties of an otherwise passive system might be destroyed in the face of certain frequency dynamics (eg: high frequency dynamics). A celebrated controversy in adaptive control [1], for instance, depended on the observation that passivity conditions normally forming part of the hypotheses of the proofs of convergence of certain adaptive control algorithms should not be assumed to be valid in practice because high frequency dynamics often neglected for modelling purposes will always be present in a real system. Failure of the passivity condition invalidated the applicability of the associated theorem on the algorithm convergence to most real-life applications and left a cloud hanging over the real-life use of the algorithm. Simulations of [1] confirmed that adverse behaviour could occur when high frequency dynamics were explicitly taken into account.

Generally speaking, a linear time-invariant (LTI) system might be called "mixed" if, over some frequencies, it is input and output strictly passive (in a sense to be made more precise later in this paper) and/or, over some frequencies, it has a gain of less than one (again, in a sense to be made more precise later); there exist no frequencies over which the system has neither of these property notions associated with it. The book [2] (see also [3] and [4]) described tools for establishing stability of adaptive systems of the type examined in [1]; that is, where passivity properties hold only for low frequency signals. Stability is established if, additionally (and in a rough manner of speaking), system gains are small at high frequencies, ie: a small gain property holds in the frequency band where the passivity condition fails.

Finite-gain stability results for "mixed" systems interconnected via a simple negative feedback loop were derived in [5], [6] (and extended to the nonlinear case in the time domain in [7], [8]) using a dissipative systems framework [9]–[14]. The objective of this paper is to now present a necessary and sufficient test for determining whether or not a given LTI system is mixed. This procedure involves the examination of the spectral characteristics of two Hamiltonian matrices, one associated with the potentially passive aspects of the system and the other associated with the notion of system small gain. Spectral conditions for positive realness of transfer function matrices are discussed in [15] and, for more general frequency domain inequalities, in [16].

Some finite-gain stability results for larger interconnections of "mixed" systems (which extend on the results for negative feedback loops obtained in [5], [6]) are also presented here. These results are derived using a dissipative systems framework, modified from [12] to allow for the frequency-dependent nature of the "mixed" systems definition and are subject to a further condition on the interconnection.

The remainder of this paper is sectioned as follows. The notion of a "mixed" system is defined further in Section II. In Section III, state-space descriptions are utilised to compute two Hamiltonian matrices and derive associated results which are required for the test of "mixedness" which is discussed in Section IV. Section V contains the finite-gain stability results for interconnections of mixed systems. Examples are provided in Section VI.

Notation

The results of this paper are concerned with LTI systems viewed in the frequency domain. We consider vector-valued frequency domain signals $f \in \mathcal{L}_2(j\mathbb{R})$, where $\mathcal{L}_2(j\mathbb{R})$ denotes the real frequency domain Lebesgue space in which

$$\|f\| = \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} f^*(j\omega) f(j\omega) d\omega \right\}^{\frac{1}{2}}$$

and the superscript $(\cdot)^*$ denotes the complex conjugate transpose. $\mathcal{L}_2(j\mathbb{R})$ is a Hilbert space under the inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} g^*(j\omega) f(j\omega) d\omega.$$

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$\mathcal{R}$ denotes the set of proper real rational transfer function matrices. For a transfer function matrix $G \in \mathcal{R}$, $G^*(s)$ is defined to mean $G(-s)^T$. $\mathcal{L}_\infty$ is a Banach space of matrix- (or scalar-) valued functions that are essentially bounded on $j\mathbb{R}$. The Hardy space, $\mathcal{H}_\infty$, is the closed subspace of $\mathcal{L}_\infty$ with functions that are analytic and bounded in the open right-half plane. In other words, $\mathcal{H}_\infty$ is the space of transfer functions of stable, LTI, continuous-time systems. $\mathcal{RH}_\infty$ denotes the subspace of $\mathcal{H}_\infty$ whose transfer function matrices are proper and real rational. The notation $A \in \mathcal{RH}_\infty^{m\times n}$ indicates such matrices with m rows and n columns.

II. Definition of a "Mixed" System

We consider a causal system with square transfer function matrix $M \in \mathcal{RH}_\infty^{m\times m}$ and denote the system's input and output signals as $e \in \mathcal{L}_2(j\mathbb{R})$ and $y \in \mathcal{L}_2(j\mathbb{R})$, respectively. We also consider a closed frequency interval $[a, b]$, where $a, b \in \mathbb{R}$.

Property 1: A causal system with transfer function matrix $M \in \mathcal{RH}_\infty^{m\times m}$ is said to be input and output strictly passive over the frequency interval $[a, b]$ if there exist $k, l > 0$ such that

$$-kM(j\omega)^*M(j\omega) + M(j\omega)^* + M(j\omega) - lI \geq 0 \qquad (1)$$

for all $\omega \in [a, b]$.

In addition, we can say that the system is input strictly passive over the frequency interval $[a, b]$ if Property 1 is satisfied with $k = 0$; output strictly passive over the frequency interval $[a, b]$ if Property 1 is satisfied with $l = 0$; and passive over the frequency interval $[a, b]$ if Property 1 is satisfied with $k = l = 0$.

Property 2: Define the system gain over the frequency interval $[a, b]$ as

$$\epsilon := \min\{\bar{\epsilon} \in \mathbb{R}_+ : -M(j\omega)^*M(j\omega) + \bar{\epsilon}^2 I \geq 0 \text{ for all } \omega \in [a, b]\}.$$

The system is said to have a gain of less than one over the frequency interval $[a, b]$ if $\epsilon < 1$.

Observe that Properties 1 and 2 require a and b to be finite. In the following, this requirement is relaxed.

Property 3: A causal system with transfer function matrix $M \in \mathcal{RH}_\infty^{m\times m}$ is said to be input and output strictly passive over the frequency interval $(-\infty, b]$, $[a, \infty)$ or $(-\infty, \infty)$ if there exist $k, l > 0$ such that (1) holds for all $\omega \in (-\infty, b]$, $[a, \infty)$ or $(-\infty, \infty)$, respectively.

Property 4: Define the system gain over the frequency interval $(-\infty, b]$, $[a, \infty)$ or $(-\infty, \infty)$ as

$$\epsilon := \inf\{\bar{\epsilon} \in \mathbb{R}_+ : -M(j\omega)^*M(j\omega) + \bar{\epsilon}^2 I \geq 0 \text{ for all } \omega \in (-\infty, b], [a, \infty) \text{ or } (-\infty, \infty), \text{ respectively}\}.$$

The system is said to have a gain of less than one over the frequency interval $(-\infty, b]$, $[a, \infty)$ or $(-\infty, \infty)$, respectively, if $\epsilon < 1$.

Motivated by the above properties, we now define a "mixed" system as follows.

Definition 5: A causal system with transfer function matrix $M \in \mathcal{RH}_\infty^{m\times m}$ is said to be "mixed" if, for each frequency $\omega \in \mathbb{R}$, either: (i) $-kM(j\omega)^*M(j\omega) + M(j\omega)^* + M(j\omega) - lI \geq 0$; or (ii) $-M(j\omega)^*M(j\omega) + \epsilon^2 I \geq 0$; or both (i) and (ii) hold. The constants $k, l > 0$ and $\epsilon < 1$ are independent of ω.

Examples of "mixed" systems include systems with the transfer functions

$$m_1(s) = \frac{3s+2}{s+5},\ m_2(s) = \frac{2s-1}{2s+4}$$

and

$$m_3(s) = \frac{3}{(s+1)(s+2)}.$$

The systems described by the transfer functions

$$m_4(s) = \frac{2s-3}{s+4}$$

and

$$m_5(s) = \frac{10}{(s+1)(s+2)}$$

are not mixed. To illustrate, consider the Nyquist diagrams of $m_2(s)$ and $m_5(s)$ as shown in Figs. 1 and 2. From Fig. 1, it is evident that there exists some frequency Ω such that the system described by the transfer function $m_2(s)$ is input and output strictly passive over $(-\infty, -\Omega]$ and $[\Omega, \infty)$ and has a gain of less than one over the frequency interval $[-\Omega, \Omega]$. For instance, one could let $\Omega = 3$. This is not the case for the system described by the transfer function $m_5(s)$.

Our aim is to provide an outline of a necessary and sufficient test for determining whether or not a given LTI system is mixed. For single-input, single-output (SISO), LTI systems, the construction of such a test is in one sense redundant as one can, for example, examine the properties of the candidate system graphically via its Nyquist plot.

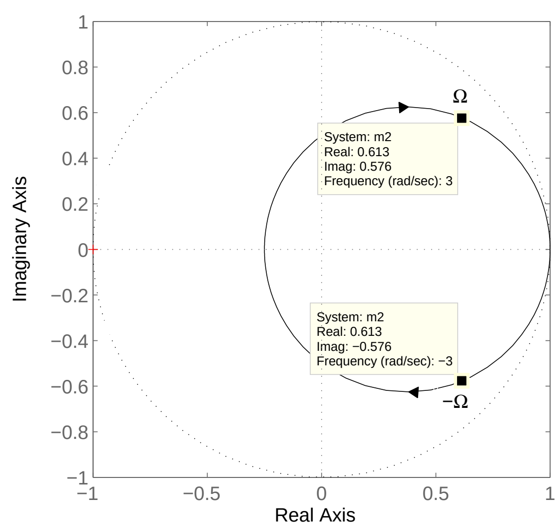

Fig. 1. Nyquist diagram of $m_2(s)$.

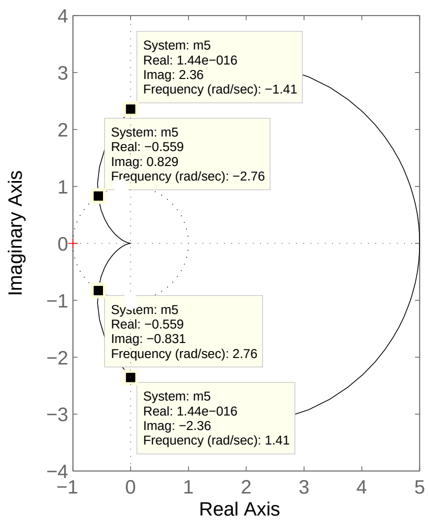

Fig. 2. Nyquist diagram of $m_5(s)$.

However, analytic tests of mixedness for multi-input, multi-output (MIMO), LTI systems (and eventually, tests for non-linear systems) are potentially more useful. In this paper, we consider the MIMO, LTI case.

III. Hamiltonian Matrices

Suppose that we are given a causal system with stable, square transfer function matrix $M = C(sI - A)^{-1}B + D$ which is described by the equations

$$\dot{x} = Ax + B\check{e},\ x(t_0) = x_0,$$
$$\check{y} = Cx + D\check{e},$$

where $x(t) \in \mathbb{R}^n$, $\check{e}(t) \in \mathbb{R}^m$ is the inverse Fourier transform of $e(j\omega)$, $\check{y}(t) \in \mathbb{R}^m$ is the inverse Fourier transform of $y(j\omega)$ and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{m\times m}$ with A Hurwitz. Suppose that the constants $k, l, \epsilon \in \mathbb{R}$ are fixed and let

$$G_1(j\omega) := -kM(j\omega)^*M(j\omega) + M(j\omega)^* + M(j\omega) - lI$$

and

$$G_2(j\omega) := -M(j\omega)^*M(j\omega) + \epsilon^2 I.$$

Assume that $X_1 := -kD^TD + D^T + D - lI$ and $X_2 := -D^TD + \epsilon^2 I$ are invertible and let $Y := I - kD$. Then, after some calculations (see [17]) which follow the technique of [15, proof of Lemma 1]), it can be shown that

$$\det(G_i(j\omega)) = \det(X_i)\det\{(j\omega I - A)^{-1}\} \det\{(j\omega I + A^T)^{-1}\}\det(j\omega I + H_i) \quad (2)$$

for $i = 1, 2$, where $H_1 :=$

$$\begin{pmatrix} -A + BX_1^{-1}Y^TC & -BX_1^{-1}B^T \\ kC^TC + C^TYX_1^{-1}Y^TC & A^T - C^TYX_1^{-1}B^T \end{pmatrix}$$

and $H_2 :=$

$$\begin{pmatrix} -A - BX_2^{-1}D^TC & -BX_2^{-1}B^T \\ C^TC + C^TDX_2^{-1}D^TC & A^T + C^TDX_2^{-1}B^T \end{pmatrix}.$$

The following result will be required for the "mixedness" test.

Lemma 6: Suppose that $k, l \in \mathbb{R}$ ($\epsilon \in \mathbb{R}$) and assume that X_1 (X_2) is invertible. The matrix $G_1(j\omega)$ ($G_2(j\omega)$) has no zero eigenvalues over $\omega \in [a, b]$ if and only if H_1 (H_2) does not have any eigenvalues on the imaginary axis between and including $-ja$ and $-jb$.

The proof of Lemma 6 (see [17]) utilises (2) and the assumptions that X_1 and X_2 are invertible and that A is Hurwitz.

IV. Testing for "Mixedness"

Given a system state-space description as described in the previous section, the goal is to determine whether or not our system is "mixed." One idea is to consider the transfer function matrix $M(s)$ that has been constructed from the state-space data and determine whether or not there exist $k, l > 0$ and $\epsilon < 1$ such that (i) and/or (ii) from Definition 5 hold for each frequency $\omega \in \mathbb{R}$. To eliminate an element of frequency-dependency from the test, however, we can utilise the state-space data directly and apply Lemma 6.

Therefore, the first step of the test is to compute H_1 and H_2 (under the assumptions that X_1 and X_2 are invertible) for some $k, l > 0$ and $\epsilon < 1$ and calculate the eigenvalues of these matrices. Existences of purely imaginary eigenvalues indicate those frequencies at which $G_1(j\omega)$ and $G_2(j\omega)$ have zero eigenvalues. Importantly, we also note the following: (i) there exist $k, l > 0$ such that $G_1(j\omega) \geq 0$ for all $\omega \in [a, b]$ if and only if $M(j\omega)^* + M(j\omega) > 0$ for all $\omega \in [a, b]$; and (ii) there exists $\epsilon < 1$ such that $G_2(j\omega) \geq 0$ for all $\omega \in [a, b]$ if and only if $-M(j\omega)^*M(j\omega) + I > 0$ for all $\omega \in [a, b]$. (Alternatively: (i) there do not exist $k, l > 0$ such that $G_1(j\omega) \geq 0$ for all $\omega \in [a, b]$ if and only if $M(j\omega)^* + M(j\omega) \ngtr 0$ for all $\omega \in [a, b]$; and (ii) there does not exist $\epsilon < 1$ such that $G_2(j\omega) \geq 0$ for all $\omega \in [a, b]$ if and only if $-M(j\omega)^*M(j\omega) + I \ngtr 0$ for all $\omega \in [a, b]$.)

This means that the constants k, l and ϵ can frequently be eliminated from the test. That is, we can often set $k = l = 0$ and $\epsilon = 1$ when applying Lemma 6; particularly, under the assumptions that $\det(M(j\infty)^* + M(j\infty)) \neq 0$ and $\det(-M(j\infty)^*M(j\infty) + I) \neq 0$, Definition 5 also becomes:

"A causal system with transfer function matrix $M \in \mathcal{RH}_\infty^{m\times m}$ is said to be "mixed" if, for each frequency $\omega \in \mathbb{R}$: either (i) $M(j\omega)^* + M(j\omega) > 0$; or (ii) $-M(j\omega)^*M(j\omega) + I > 0$; or both (i) and (ii) hold."

If there exist frequencies at which $M(j\omega)^* + M(j\omega)$ has zero eigenvalues then we divide the frequency range $(-\infty, \infty)$ up into intervals with the frequencies corresponding to the zero eigenvalues as the interval endpoints. (If there exist no frequencies at which $M(j\omega)^* + M(j\omega)$ has zero eigenvalues then we leave the frequency range $(-\infty, \infty)$ intact and think of it as a single "division.") Similarly, a separate set of divisions of the entire frequency range $(-\infty, \infty)$ can be made based on the frequencies at which $-M(j\omega)^*M(j\omega) + I$ has zero eigenvalues. (Again, the frequency range $(-\infty, \infty)$ is left intact if there exist no frequencies at which $-M(j\omega)^*M(j\omega) + I$ has zero eigenvalues.) We now have two different sets of frequency range divisions: Set of Divisions 1 and Set of Divisions 2.

Finally, we check the sign definiteness of the matrix $M(j\omega)^* + M(j\omega)$ over each interval belonging to Set of Divisions 1 and the sign definiteness of the matrix $-M(j\omega)^*M(j\omega) + I$ over each interval belonging to Set of Divisions 2. Testing at one frequency (eg: at the mid-point) per interval is sufficient. Those intervals over which $M(j\omega)^* + M(j\omega) > 0$ and those intervals over which $-M(j\omega)^*M(j\omega) + I > 0$ are identified and we determine whether or not there exists some combination of these intervals that span the entire frequency range; that is, if $M(j\omega)^* + M(j\omega) > 0$ and/or $-M(j\omega)^*M(j\omega) + I > 0$ for each $\omega \in \mathbb{R}$ then the system is "mixed." If, for some $\omega \in \mathbb{R}$, either $M(j\omega)^* + M(j\omega) \not> 0$ or $-M(j\omega)^*M(j\omega) + I \not> 0$ then the system is not "mixed."

V. "Mixed" Systems Interconnections

Once it is possible to determine whether or not systems are mixed, finite-gain stability results may be called upon to examine the properties of interconnections of such systems. Suppose that $\alpha(\omega)$ is an arbitrary, real, continuous, (even) function of frequency and, moreover, that $0 \leq \alpha(\omega) \leq 1$.

Lemma 7: Suppose that a causal system with transfer function matrix $M \in \mathcal{RH}_\infty^{m\times m}$ is "mixed." Then there exists an $\alpha(\omega)$ (as described above) such that

$$\alpha(\omega)[-kM(j\omega)^*M(j\omega) + M(j\omega)^* + M(j\omega) - lI] + (1-\alpha(\omega))[-M(j\omega)^*M(j\omega) + \epsilon^2 I] \geq 0$$

for all $\omega \in \mathbb{R}$.

The proof of Lemma 7 follows from the fact that the sum of two $m\times m$ positive semi-definite matrices is positive semi-definite, as is any convex combination of two such matrices [18, page 258]. At frequencies at which $G_1(j\omega)$ or $G_2(j\omega)$ are not positive semi-definite, one can set $\alpha(\omega)$ or $1-\alpha(\omega)$, respectively, to 0 or 1, respectively.

Remark 8: If $M \in \mathcal{RH}_\infty$ (ie: the system is SISO), then the condition in Lemma 7 is an if and only if statement.

Now consider a linear interconnection of N mixed subsystems (the precise form of the interconnection to be described in a moment). We integrate each of the inequalities

$$e_i(j\omega)^*\{\alpha(\omega)[-k_iM_i(j\omega)^*M_i(j\omega) + M_i(j\omega)^* + M_i(j\omega) - l_iI] + (1-\alpha(\omega))[-M_i(j\omega)^*M_i(j\omega) + \epsilon_i^2 I]\}e_i(j\omega) \geq 0,$$

where $i = 1,\ldots,N$ and $e_i \in \mathcal{L}_2(j\mathbb{R})$ is the input to subsystem i, with respect to ω and multiply each integral by $\frac{1}{2\pi}$ to obtain the following condition on the interconnection: there exists an $\alpha(\omega)$ such that

$$\langle y_i, q_i(\omega)y_i\rangle + 2\langle y_i, s_i(\omega)e_i\rangle + \langle e_i, r_i(\omega)e_i\rangle \geq 0 \qquad (3)$$

for all $e_i \in \mathcal{L}_2(j\mathbb{R})$, for each $i = 1,\ldots,N$, where

$$q_i(\omega) := -(k_i\alpha(\omega) + 1 - \alpha(\omega))$$
$$s_i(\omega) := \alpha(\omega)$$
$$r_i(\omega) := \epsilon_i^2(1-\alpha(\omega)) - l_i\alpha(\omega)$$

and $k_i, l_i > 0$ and $\epsilon_i < 1$. The interconnection is described by

$$e_i = u_i - \sum_{j=1}^{N} H_{ij}y_j, \; i = 1,\ldots,N,$$

where $y_i \in \mathcal{L}_2(j\mathbb{R})$ is the output of subsystem i, $u_i \in \mathcal{L}_2(j\mathbb{R})$ is an external input and H_{ij} is a constant matrix. Writing

$$\boldsymbol{e} := \begin{pmatrix} e_1 \\ \vdots \\ e_N \end{pmatrix}, \boldsymbol{y} := \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix} \text{ and } \boldsymbol{u} := \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix},$$

the interconnection description may be written more compactly as

$$\boldsymbol{e} = \boldsymbol{u} - H\boldsymbol{y}, \qquad (4)$$

where H is a matrix with block entries H_{ij}. Denote $\tilde{Q} :=$ diag$(q_1(\omega)I,\ldots,q_N(\omega)I)$, $\tilde{S} :=$ diag$(s_1(\omega)I,\ldots,s_N(\omega)I)$ and $\tilde{R} :=$ diag$(r_1(\omega)I,\ldots,r_N(\omega)I)$ and let

$$\bar{Q} := \tilde{Q} + H^T\tilde{R}H - \tilde{S}H - H^T\tilde{S},$$

noting that $\bar{Q}^T = \bar{Q}$. We have the following result, modified to allow for frequency-dependent $\tilde{Q}$, $\tilde{S}$ and $\tilde{R}$, from [12, Theorem 1].

Theorem 9: An interconnection of "mixed" subsystems, with input $\boldsymbol{u}$ and output $\boldsymbol{y}$, as described above, is finite-gain stable if $\bar{Q}$ is negative definite.

Refer to [17] for the proof of Theorem 9. In [5, Theorem 6] and [6, Theorem 1], it was shown that, for

$$H = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

the matrix $\bar{Q}$ is guaranteed to be negative definite and thus the interconnection, depicted in Fig. 3, is always finite-gain stable. This is a generalisation of the small gain theorem and the passivity theorem, albeit here, restricted to LTI systems.

Let us now denote $K :=$ diag$(k_1I,\ldots,k_NI)$, $L :=$ diag$(l_1I,\ldots,l_NI)$ and $E :=$ diag$(\epsilon_1^2I,\ldots,\epsilon_N^2I)$. The following result is an alternative sufficient condition for finite-gain stability. The condition is frequency-independent (see [17] for the proof).

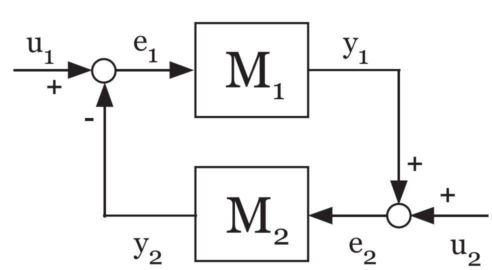

Fig. 3. A negative feedback interconnection.

Theorem 10: An interconnection of "mixed" subsystems, with input $\boldsymbol{u}$ and output $\boldsymbol{y}$, as described above, is finite-gain stable if there exists a positive definite matrix $P = \text{diag}(p_1 I, \dots, p_N I)$ such that $PK + PH + H^T P + H^T PLH > 0$ and $P - H^T PEH > 0$.

Remark 11: Finite-gain stability of the interconnection, depicted in Fig. 3, is guaranteed via the alternative sufficient condition presented in Theorem 10 as there exists a $P = \text{diag}(p_1 I, p_2 I) > 0$ such that

$$\begin{pmatrix} (p_1 k_1 + p_2 l_2)I & (p_1 - p_2)I \\ (p_1 - p_2)I & (p_1 l_1 + p_2 k_2)I \end{pmatrix} > 0$$

and

$$\begin{pmatrix} (p_1 - p_2 \epsilon_2^2)I & 0 \\ 0 & (p_2 - p_1 \epsilon_1^2)I \end{pmatrix} > 0.$$

For instance, setting $p_1 = p_2$, the condition reduces to

$$\begin{pmatrix} (k_1 + l_2)I & 0 \\ 0 & (l_1 + k_2)I \end{pmatrix} > 0$$

and

$$\begin{pmatrix} (1 - \epsilon_2^2)I & 0 \\ 0 & (1 - \epsilon_1^2)I \end{pmatrix} > 0$$

which is satisfied since $k_i, l_i > 0$ and $\epsilon_i < 1$ for $i = 1, 2$.

VI. Examples

The following two examples illustrate various aspects of the test for mixedness.

Example 1: (SISO "mixed" system) Given the state-space data $A = -2$, $B = 2$, $C = -1.75$ and $D = 1.5$ from which the transfer function

$$m_6(s) = \frac{3s - 1}{2s + 4}$$

can be constructed, and setting $k = l = 0$ and $\epsilon = 1$, we get

$$H_1 = \begin{pmatrix} 0.8\dot{3} & -1.\dot{3} \\ 1.02083\dot{3} & -0.8\dot{3} \end{pmatrix}$$

and

$$H_2 = \begin{pmatrix} -2.2 & 3.2 \\ -2.45 & 2.2 \end{pmatrix}$$

(noting that $m_6(j\infty)^* + m_6(j\infty) \neq 0$ and $m_6(j\infty)^* m_6(j\infty) \neq 1$). The matrix H_1 has two purely imaginary eigenvalues, $\pm 0.8165i$. Breaking the frequency range $(-\infty, \infty)$ up into the intervals $(-\infty, -0.8165]$, $[-0.8165, 0.8165]$ and $[0.8165, \infty)$ and examining the sign definiteness of $m_6(j\omega)^* + m_6(j\omega)$ at a single frequency point from the interiors of each of these intervals (eg: at $\omega = -1, 0, 1$) yields $m_6(-j1)^* + m_6(-j1) > 0$, $m_6(j0)^* + m_6(j0) \ngtr 0$ and $m_6(j1)^* + m_6(j1) > 0$. Thus, the system is passive over $(-\infty, -0.8165]$ and $[0.8165, \infty)$ and a system gain of less than one over $[-0.8165, 0.8165]$ is required in order for it to be "mixed."

The matrix H_2 has two purely imaginary eigenvalues, $\pm 1.732i$. Observing the sign definiteness of $-m_6(j\omega)^* m_6(j\omega) + 1$ at a single frequency point from the interiors of each of the intervals $(-\infty, -1.732]$, $[-1.732, 1.732]$ and $[1.732, \infty)$ (eg: at $\omega = -2, 0, 2$) yields $-m_6(-j2)^* m_6(-j2) + 1 \ngtr 0$, $-m_6(j0)^* m_6(j0) + 1 > 0$ and $-m_6(j2)^* m_6(j2) + 1 \ngtr 0$. The system has a gain of one over the frequency interval $[-1.732, 1.732]$ and $[-0.8165, 0.8165]$ is a subset of this interval. Thus, the system is "mixed." See Fig. 4 for an illustration of the system's frequency response.

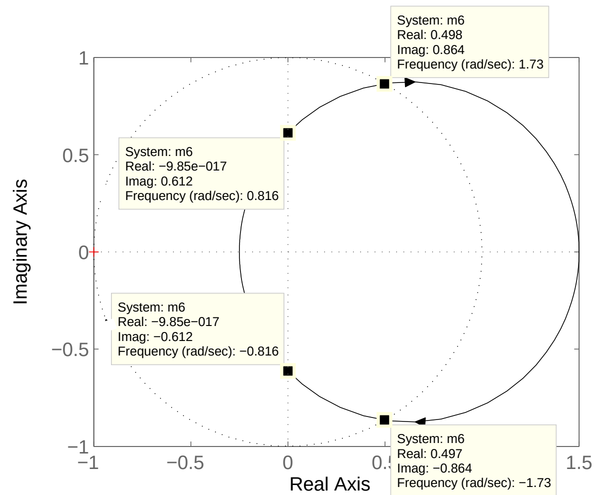

Fig. 4. Nyquist diagram of $m_6(s)$.

Example 2: (MIMO system, not "mixed") Given the state-space data $A =$ [-3 -2 0 0 0 0; 1 0 0 0 0 0; 0 0 -5 0 0 0; 0 0 0 -7 -3 0; 0 0 0 4 0 0; 0 0 0 0 0 -1], $B =$ [2 0; 0 0; 4 0; 0 2; 0 0; 0 1], $C =$ [0 1.5 0 0 1.625 0; 0 0 -3.25 0 0 1] and $D =$ [0 0; 3 0] from which the transfer function matrix

$$M_7(s) = \begin{pmatrix} \frac{3}{(s+1)(s+2)} & \frac{13}{(s+3)(s+4)} \\ \frac{3s+2}{s+5} & \frac{1}{s+1} \end{pmatrix}$$

may be constructed, and setting $k = l = 0$ and $\epsilon = 1$, we obtain H_1 and H_2 (see Fig. 5) noting that $\det(M_7(j\infty)^* + M_7(j\infty)) \neq 0$ and $\det(-M_7(j\infty)^* M_7(j\infty) + I) \neq 0$. The matrix H_1 has two purely imaginary eigenvalues, $\pm 0.5959i$. Breaking the frequency range $(-\infty, \infty)$ up into the intervals $(-\infty, -0.5959]$, $[-0.5959, 0.5959]$ and $[0.5959, \infty)$ and examining the sign definiteness of $M_7(j\omega)^* + M_7(j\omega)$ at a single frequency point from the interiors of each of these intervals (eg: at $\omega = -1, 0, 1$) yields $M_7(-j1)^* + M_7(-j1) \ngtr 0$, $M_7(j0)^* + M_7(j0) > 0$ and $M_7(j1)^* + M_7(j1) \ngtr 0$. Thus, the system is passive over $[-0.5959, 0.5959]$ and a system gain of less than one over $(-\infty, -0.5959]$ and $[0.5959, \infty)$ is required in order for it to be "mixed."

The matrix H_2 does not have any purely imaginary eigenvalues which means that the sign definiteness of $-M_7(j\omega)^* M_7(j\omega) + I$ will remain the same over the entire frequency range $(-\infty, \infty)$. Since $-M_7(j0)^* M_7(j0) + I$ is an indefinite matrix, the system does not have a gain of less than one over $(-\infty, \infty)$ and is hence not "mixed."

VII. Conclusions and Future Work

A test for determining whether or not a causal, stable, MIMO, LTI system is "mixed" was developed. Implemen-

$$H_1 = \begin{pmatrix} 3 & 2 & -2.1\dot{6} & 0 & 0 & 0.\dot{6} & 0 & 0 & 0 & -1.\dot{3} & 0 & -0.\dot{6} \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.\dot{6} & 0 & 0 & 1.\dot{3} & 0 & 0 & 0 & -2.\dot{6} & 0 & -1.\dot{3} \\ 0 & 1 & 0 & 7 & 4.08\dot{3} & 0 & -1.\dot{3} & 0 & -2.\dot{6} & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0.541\dot{6} & 1 & -0.\dot{6} & 0 & -1.\dot{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1.625 & 0 & 0 & 0.5 & -2 & 0 & 0 & -1 & 0 & -0.5 \\ 0 & -1.625 & 0 & 0 & -1.76041\dot{6} & 0 & 2.1\dot{6} & 0 & -0.\dot{6} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -7 & 4 & 0 \\ 0 & 0 & -1.76041\dot{6} & 0 & 0 & 0.541\dot{6} & 0 & 0 & 0 & -4.08\dot{3} & 0 & -0.541\dot{6} \\ 0 & 0.5 & 0 & 0 & 0.541\dot{6} & 0 & -0.\dot{6} & 0 & -1.\dot{3} & 0 & 0 & -1 \end{pmatrix} \text{ and}$$

$$H_2 = \begin{pmatrix} 3 & 2 & -2.4375 & 0 & 0 & 0.75 & 0.5 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.125 & 0 & 0 & 1.5 & 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7 & 3 & 0 & 0 & 0 & 0 & -4 & 0 & -2 \\ 0 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2.25 & 0 & 0 & 2.4375 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1.3203125 & 0 & 0 & 0.40625 & 2.4375 & 0 & -0.125 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -7 & 4 & 0 \\ 0 & 2.4375 & 0 & 0 & 2.640625 & 0 & 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0.40625 & 0 & 0 & -0.125 & -0.75 & 0 & -1.5 & 0 & 0 & -1 \end{pmatrix}$$

Fig. 5. The Hamiltonian matrices.

tation of the test requires little more than determining the purely imaginary eigenvalues of two Hamiltonian matrices. Once "mixedness" is determined, finite-gain stability results for interconnections of such systems may be implemented provided that the required conditions on the interconnection are met. The case of strictly proper systems is to be dealt with in a future publication (see [17]).

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